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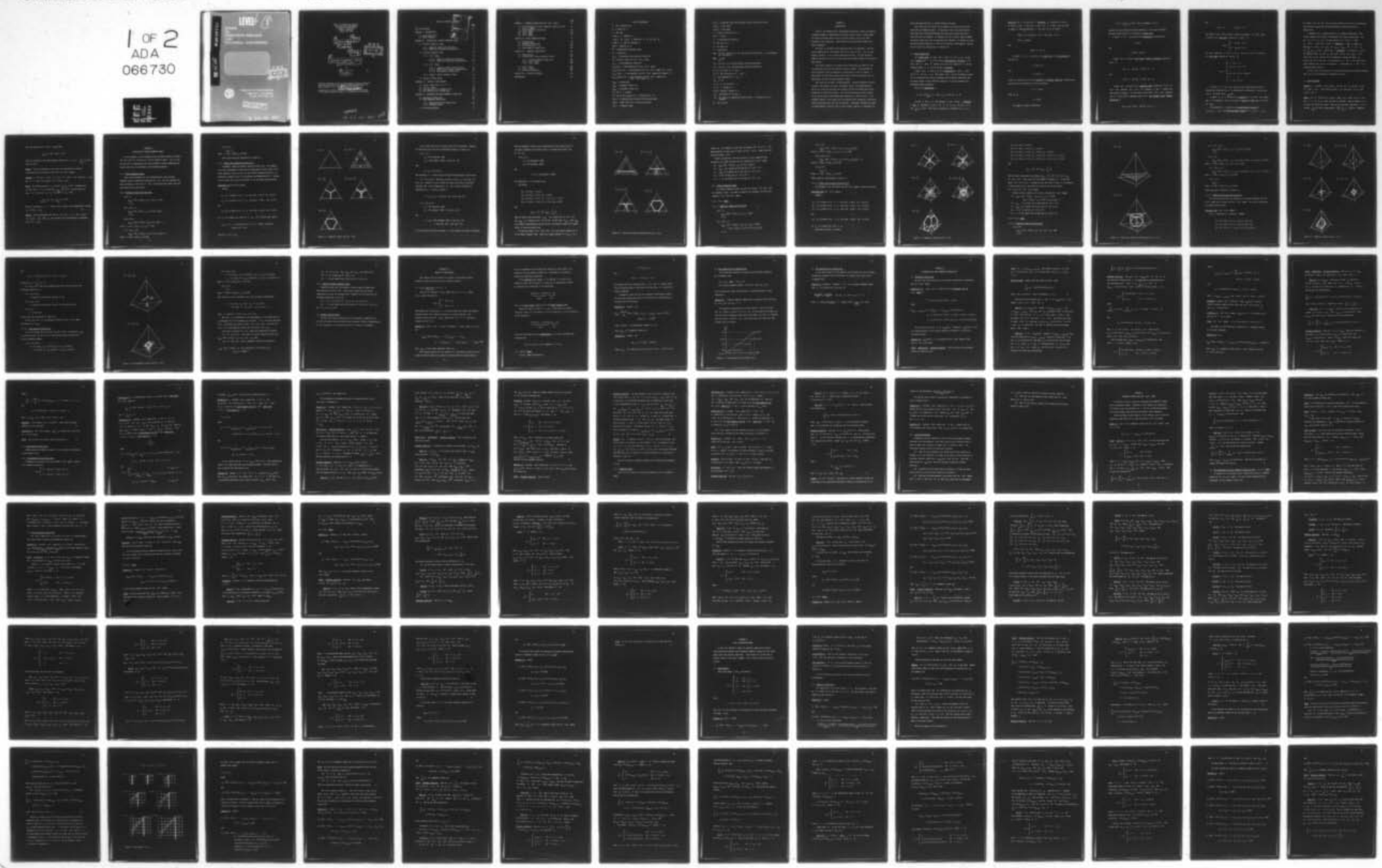

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ITHACA, NEW YORK

9 TECHNICAL REPORT NO. 387 14 TR-

11 August 1978

12 143p.

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STABLE SETS FOR SYMMETRIC, n-PERSON COOPERATIVE GAMES

BY

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Research supported in part by the National Science Foundation under grants MPS75-02024 and MCS77-28392 and the Office of Naval Research under contract N00014-75-C-0678.

15 NSF-MPS75-02024

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LIST OF NOTATIONS

A: set of imputations

B: a subset of A

C: the core

x dom y: x dominates y

x dom y via $S_x | T_y$: x dominates y via S_x over T_y.

x d$\not{o}$m y: x does not dominate y

Dom B: dominion of B

E^n: n-dimensional Euclidean space

K: a stable set

$K_{e,d}$: an extremely discriminatory stable set

K_h: the Hart stable set for $(n;k)_h$ games

$K_{s,s}$: a semi-symmetric stable set

$K_{s,s}(n;k)$: a semi-symmetric set for (n;k) games

$K_{s,s}(n;(n+1)/2)$: a semi-symmetric set for (n;k) games with $n = 2k-1$

$K_{s,s,\omega}(N,k)$: a semi-symmetric set for (N;k) games with respect to ω

$K_{s,s,\omega}(N,(n+1)/2)$: a semi-symmetric set for (N,k) games with $n = 2k-1$ with respect to ω

K_{sub}: a subsolution

K_{sym}: a symmetric stable set

K_{sys}: a systematic stable set

N: set of n players

2^N: set of all subsets of N (coalitions of N)

(N,v): a characteristic function form n-person game

(N,k): a game with vital k-person coalitions

(n,v): a symmetric game

$(n;k)$: a symmetric game with strongly vital k-person coalitions

$(n;k)_b$: a Bott game

$(n;k)_h$: a Hart game

S: a subset (coalition) of N

U(B): A - Dom B

v: a characteristic function

$\bar{v}$: the cover of v

$\emptyset$: the empty set

ω: a semi-quota

$\langle x \rangle$: the set consisting of all imputations obtained from x by permuting its coordinates

$\langle B \rangle$: $\bigcup_{x \in B} \langle x \rangle$

$\lceil A \rceil$: the set of all nonincreasingly ordered imputations

$\lfloor A \rfloor$: the set of all nondecreasingly ordered imputations

$B \cup D$: the union of B and D

$B \cap D$: the intersection of B and D

B^c: the complement of B in A

B-D: $B \cap D^c$

$B \subseteq D$: B is included in D

$x \in B$: x belongs to B

$[[p]]$: greatest integer in p

$|S|$: cardinality of the set S

$\binom{m}{n}$: the number of combinations which choose n elements out of m elements

$\Box$: end of proof

CHAPTER I

INTRODUCTION

Since J. von Neumann and O. Morgenstern presented a theory of n-person cooperative games in characteristic function form in 1944, a large number of studies have been made on their solution concept, called the vN-M solution or the stable set. These works can be classified roughly into three categories.

The first is concerned with questions about its existence. This was solved negatively for the general case by W. Lucas in 1967. His counter-example, however, is of a rather specialized nature. So this existence problem continues as one of the most important research areas in cooperative game theory.

The second category is concerned with determining the explicit form of particular solutions for special classes of games. This approach is quite important from the viewpoint of application as well as theory. A good number of interesting results have been obtained along this line, in particular, for the so-called symmetric games.

The third one is about its modifications. Several different solution concepts; for example the core, the Shapley value, the bargaining sets and so on, have been proposed and studied; as well as several more direct variations of the vN-M solution. One recently proposed solution concept of the latter type is the subsolution defined by A. Roth in 1976. It is somewhat similar to the stable set and moreover Roth succeeded in establishing its existence when the core is nonempty. Additional analysis for games in characteristic function form is still needed, however, to determine the

nature and applicability of various solution concepts.

This study will be devoted to an analysis of stable sets and subsolutions for symmetric games. In Chapters II and III several results which have been obtained previously and which are closely related to this work will be reviewed. In Chapters IV and V several types of games and their stable sets will be presented and analyzed. In Chapter VI, some production game defined by S. Hart will be further investigated. Finally, we will deal with subsolutions in Chapter VII.

1.1 Basic Definitions

An n-person game is a pair (N,v) where $N = \{1,2,\ldots,n\}$ is the set of players and v is a real-valued characteristic function on 2^N with $v(\emptyset) = 0$. Here 2^N denotes the set of all subsets of N and any nonempty subset of N will be called a coalition.

A game (N,v) is said to be (0,1)-normalized if $v(N) = 1$ and $v(\{i\}) = 0$ for all $i \in N$. Most games (N,v) can be converted to their (0,1)-normalized form without changing their essential structure, nor the basic nature of most solution concepts. So we will assume (0,1)-normalized games throughout.

The set of imputations is

$$A = \{x \in E^n \mid \sum_{i \in N} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \in N\}.$$

For any x and $y \in A$ and nonempty $S \subseteq N$, we say x dominates y via S, denoted by $x \operatorname{dom} y$ via S, if $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leq v(S)$. This latter inequality is referred to as S is

effective for x. We also say x dominates y, denoted by x dom y, if there is some S such that x dom y via S. x d$\not{o}$m y will be used to imply x does not dominate y. For any $B \subseteq A$ we define

$$\mathrm{Dom}_S B = \{y \in A \mid x \text{ dom } y \text{ via } S \text{ for some } x \in B\}$$

$$\mathrm{Dom}\, B = \bigcup_S \mathrm{Dom}_S B$$

and

$$U(B) = A - \mathrm{Dom}\, B.$$

A subset K of A is said to be a stable set or a vN-M solution if and only if

$$K \cap \mathrm{Dom}\, K = \emptyset$$

and

$$K \cup \mathrm{Dom}\, K = A.$$

These two conditions are called internal and external stability, respectively; and they can be expressed as the one condition

$$K = A - \mathrm{Dom}\, K$$

that is, as

$$K = U(K).$$

The core of a game is defined by

$$C = \{x \in A \mid \sum_{i \in S} x_i \geq v(S) \text{ for all nonempty } S \subseteq N\}.$$

Clearly the core satisfies internal stability. If it also satisfies external stability, then it is called the <u>stable</u> <u>core</u>.

A subset L of A is said to be a <u>subsolution</u> if and only if

$$L \subseteq U(L)$$

and

$$L = U^2(L) = U(U(L)).$$

A game (N,v) is said to <u>have</u> <u>vital</u> k-<u>person</u> <u>coalitions</u>, denoted by (N,k) if

$$v(S) > 0 \quad \text{for all } S \text{ with } |S| = k$$

and

$$v(S) = 0 \quad \text{for all } S \text{ with } |S| < k.$$

A game (N,v) is said to be a <u>symmetric</u> <u>game</u>, denoted by (n,v), if $v(S) = v(T)$ whenever $|S| = |T|$, i.e., whenever S and T contain the same number of players. In this case we also write $v(S) = v(s)$ whenever $|S| = s$. We say that a symmetric game (n,v) <u>has</u> <u>strongly</u> <u>vital</u> k-<u>person</u> <u>coalitions</u> if

$$v(s) \leq v(k) \cdot (s/k) \quad \text{for all } k \leq s < n$$

and

$$v(s) = 0 \qquad \text{for all } s < k.$$

The symbol $(n;k)$ will be used to denote such games. An $(n;k)$ game is said to be a <u>Bott game</u>, denoted by $(n;k)_b$, if

$$v(s) = \begin{cases} 1 & \text{for all } s \geq k \\ 0 & \text{for all } s < k. \end{cases}$$

An $(n;k)$ game with $n = qk + r$ $(q \geq 2$ and $0 \leq r \leq k-1)$ is said to be a <u>Hart game</u>, denoted by $(n;k)_h$, if

$$v(s) = \begin{cases} 0 & \text{for } s < k \\ 1/q & \text{for } k \leq s < 2k \\ \dots & \dots \\ j/q & \text{for } jk \leq s < (j+1)k \\ \dots & \dots \\ 1 & \text{for } qk \leq s. \end{cases}$$

For any $x \in A$, let $\langle x \rangle$ denote the set which consists of all imputations obtained from x by permuting its coordinates. And for any $B \subseteq A$, we define $\langle B \rangle = \bigcup_{x \in B} \langle x \rangle$.

A subset B of A is said to be <u>symmetric</u> if $B = \langle B \rangle$. If a stable set K is symmetric, then it is called a <u>symmetric stable set</u> and denoted by K_{sym}.

An imputation x is said to be <u>nonincreasingly ordered</u> if $x_1 \geq x_2 \geq \dots \geq x_n$ and <u>nondecreasingly ordered</u> if $x_1 \leq x_2 \leq \dots \leq x_n$.

The symbol $\lceil A \rceil$ and $\lfloor A \rfloor$ will be used to denote the set of all nonincreasingly ordered imputations and nondecreasingly ordered imputations, respectively.

A symmetric set is characterized by its ordered imputations. Thus we will redefine the concept of domination for ordered imputations. For any $x,y \in \lceil A \rceil$ (or $\lfloor A \rfloor$) and any nonempty $S_x = \{i(1),\ldots,i(m)\}$, $T_y = \{j(1),\ldots,j(m)\} \subseteq N$, we say x dominates y via S_x over T_y, denoted by x dom y via $S_x|T_y$ if $x_{i(r)} > y_{j(r)}$ for $r = 1,2,\ldots,m$ and $\sum_{r=1}^{m} x_{i(r)} \leq v(S_x)$. And we say x dominates y, denoted by x dom y, if there are some S_x and T_y such that x dom y via $S_x|T_y$. It is clear that we can, without loss of generality, assume the above T_y denotes the set of the last m coordinates if x and y are nonincreasingly ordered, and the set of the first m coordinates if x and y are nondecreasingly ordered.

We will close this chapter by stating and proving the following theorems which will be used implicitly throughout this work.

1.2 Basic Theorems

Theorem 1.1: Consider $(n;k)$ games. For any $x,y \in A$ and any $T \subseteq N$, if x dom y via T then there is some $S \subseteq N$ such that $|S| = k$ and x dom y via S.

Proof: Let $T = \{i(1),i(2),\ldots,i(m)\}$. Then $m \geq k$ since $v(s) = 0$ for all $s < k$. If $m = k$, then no proof is required. Thus we assume $m > k$ and that the theorem is false. Then for all k-person coalitions S in T, we have $\sum_{i\in S} x_i > v(k)$ which implies $\binom{m-1}{k-1} \sum_{i\in T} x_i > \binom{m}{k} v(k)$. Together

with the definition of $(n;k)$, we must have

$$\sum_{i\in T} x_i > v(k)\cdot(m/k) > v(m)$$

which is contrary to the effectiveness condition on T, i.e., $\sum_{i\in T} x_i \leq v(T)$ fails to hold. $\square$

<u>Remark</u>: From this theorem, we only need to concentrate on k-person coalitions when we determine stable sets for $(n;k)$ games.

<u>Theorem</u> 1.2: For any x and y of $\lceil A \rceil$, if x dom y via $S_x | \{n-m+1,\ldots,n\}_y$ and $x \in C$ then we can assume $S_x = \{n-m+1,\ldots,n\}_x$.

<u>Proof</u>: The effectiveness of S_x implies $\sum_{i\in S} x_i \leq v(m)$. Together with the fact that $x \in C$, we must have $\sum_{i\in S_x} x_i = v(m)$. Suppose there is some $j \in N$ such that $x_j < x_\ell$ for some $\ell \in S_x$. Then

$$\sum_{i\in S_x-\{\ell\}} x_i + x_j < \sum_{i\in S_x} x_i = v(m)$$

which is contrary to $x \in C$. Thus we can, without loss of generality, assume $S_x = \{n-m+1,\ldots,n\}_x$. $\square$

<u>Remark</u>: One can similarly show that if $x,y \in \lfloor A \rfloor$, $x \in C$ and x dom y via $S_x | \{1,\ldots,m\}_y$ then S_x can, without loss of generality, be assumed to be $\{1,\ldots,m\}_x$.

CHAPTER II

3-PERSON AND 4-PERSON SYMMETRIC GAMES

In this chapter, we will briefly review the known results on stable sets and cores for 3-person and 4-person symmetric games. This is done for the sake of completeness and in an attempt to better understand the results which will be obtained in the following chapters.

2.1 3-Person Symmetric Games

Since we are assuming the (0,1)-normalization, each 3-person symmetric game is completely determined by $v(2)$. We will classify the cases according to the value of $v(2)$ and describe what stable sets and cores look like for each case.

2.1.1 Symmetric Stable Sets and Cores

$v(2) = 1$:

$$K_{sym} = \langle\{x \in \lceil A\rceil \mid x_1 = x_2 = 1/2,\ x_3 = 0\}\rangle.$$

$$C = \emptyset.$$

$2/3 < v(2) < 1$:

$$K_{sym} = \langle\{x \in \lceil A\rceil \mid x_1 = x_2 \geq 1\text{-}v(2) \geq x_3\}\rangle.$$

$$C = \emptyset.$$

$v(2) = 2/3$:

$$K_{sym} = \langle\{x \in \lceil A\rceil \mid x_1 = x_2 \geq 1/3 \geq x_3\}\rangle \cup C$$

where $C = \langle\{x \in \lceil A\rceil \mid x_1 = x_2 = x_3 = 1/3\}\rangle$.

$1/2 < v(2) < 2/3$:

$$K_{sym} = \langle\{x \in \lceil A\rceil \mid x_1 = x_2 \geq 1\text{-}v(2) \geq x_3\}\rangle \cup C$$

where $C = \langle\{x \in \lceil A\rceil \mid x_1 \leq 1\text{-}v(2)\}\rangle$.

$v(2) \leq 1/2$:

$$K_{sym} = C$$

where $C = \langle\{x \in \lceil A \rceil | x_1 \leq 1-v(2)\}\rangle$.

These five cases are illustrated in Figure 2.1

2.1.2 Stable Sets Obtained Systematically

In general, there are several types of stable sets. For example, we can get nonsymmetric stable sets by replacing the three lines in the above symmetric stable sets by the well-known "bargaining curves", as indicated on pages 403-419 of von Neumann and Morgenstern [37]. Another type of stable sets could be obtained in the following systematic way.

Systematic way (for (3;2) games):

Define

$$A_1 = \{x \in A | \text{there is no } y \in A \text{ such that } y \text{ dom } x \text{ via } \{2,3\}\},$$

$$A_2 = \{x \in A_1 | \text{there is no } y \in A_1 \text{ such that } y \text{ dom } x \text{ via } \{1,3\}\}$$

and

$$A_3 = \{x \in A_2 | \text{there is no } y \in A_2 \text{ such that } y \text{ dom } x \text{ via } \{1,2\}\}.$$

If A_3 is a stable set, then let $K = A_3$. If it is not, then define

$$A_{3,1} = \{x \in A - \text{Dom } A_3 | \text{there is no } y \in A - \text{Dom } A_3 \text{ such that } y \text{ dom } x \text{ via } \{1,2\}\}$$

and let $K = A_3 \cup A_{3,1}$.

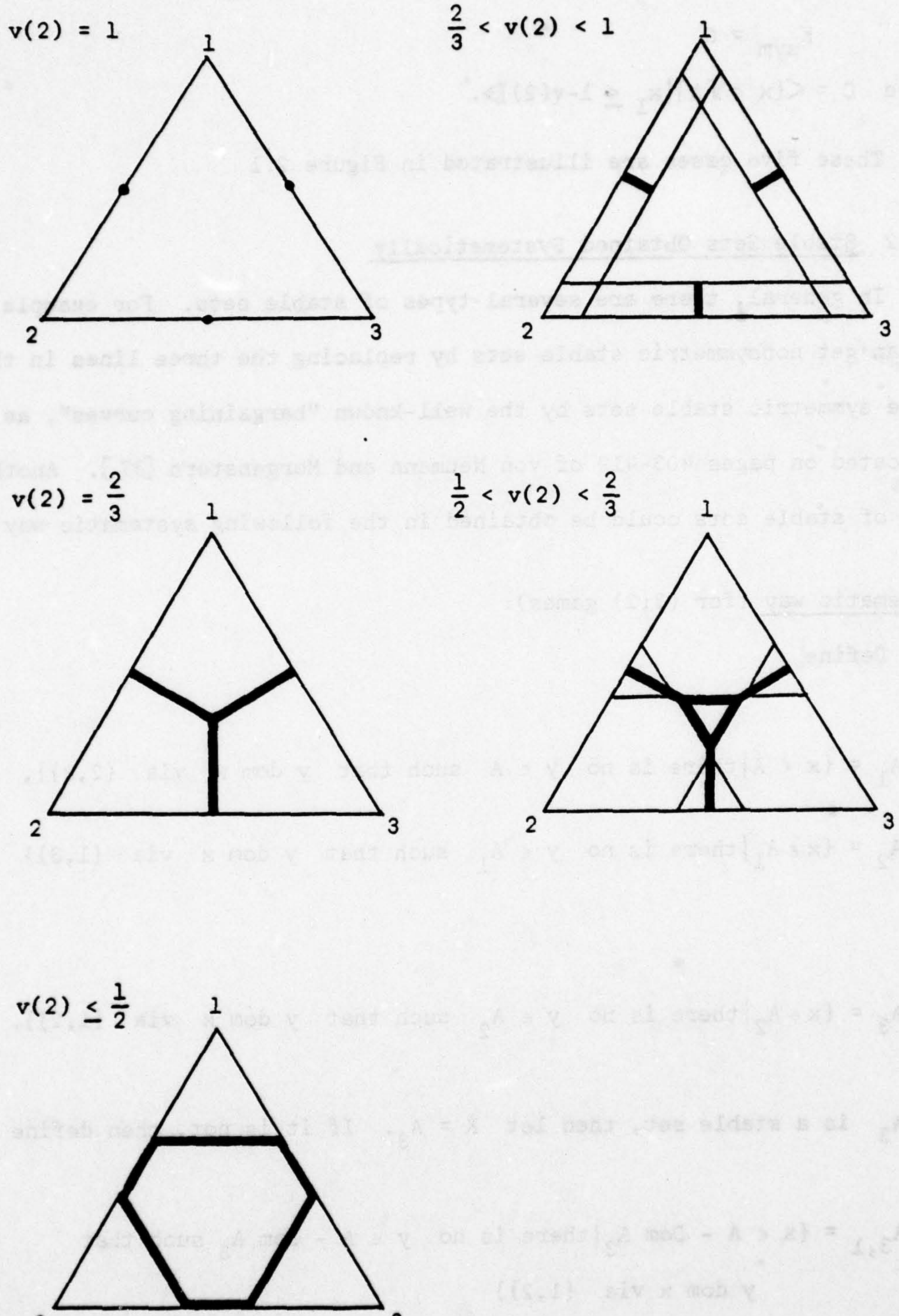


Figure 2.1 Symmetric stable sets for (3;2)

Let us check what type of stable sets will be obtained. Consider the following cases and the corresponding figures in Figure 2.2.

$v(2) > 3/4$:

A_1 = the trapezoid ABCD

A_2 = (the diamond EFDB) $\cup$ (the line CE)

and

$K = A_3$ = the line CD.

The resulting K is one of the so-called "discriminatory" stable sets.

$2/3 < v(2) \leq 3/4$: Similarly as above, we get A_3 = the line CD. This A_3, however, is not a stable set since any point in the small triangle GHI is not dominated by A_3. But a second iteration on imputations of $A - \text{Dom}\ A_3$ produces

$K = A_3 \cup A_{3,1}$ = (the line CD) $\cup$ (the line GH).

$1/2 < v(2) \leq 2/3$:

A_1 = the trapezoid ABCD,

A_2 = (the diamond EFDB) $\cup$ (the line CE)

and

$K = A_3$ = (the triangle EGH) $\cup$ (the line CE) $\cup$ (the line HF) $\cup$ (the line GD).

It is well-known that the resulting K is the stable set which is obtained

from the symmetric stable set by replacing the three middle lines in the small triangles by the three lines, or "bargaining curves", CE, HF and GD.

$v(2) \leq 1/2$:

A_1 = the trapezoid ABCD,

A_2 = the pentagon EFBDC

and

$$K = A_3 = \text{the hexagon EFHGDC.}$$

The resulting K is the stable core.

Now define

$$A_1' = \{x \in A \mid x_1 = 1-v(2)\},$$
$$A_2' = \{x \in A \mid x_1 < 1-v(2),\ x_2 = 1-v(2)\}$$
$$A_3' = \{x \in A \mid x_1 < 1-v(2),\ x_2 < 1-v(2),\ x_3 = 1-v(2)\}$$
$$A_4' = \{x \in A \mid x_1 < 1-v(2),\ x_2 < 1-v(2),\ x_3 < 1-v(2)\}$$

and

$$K_{e.d.} = A_1' \text{ and } K_{sys} = \bigcup_{i=1}^{4} A_i'.$$

Then as easily seen from above, $K_{e.d.}$ is a stable set if $v(2) > 3/4$ and K_{sys} is a stable set if $v(2) \leq 2/3$. We will call $K_{e.d.}$ and K_{sys} an extremely discriminatory stable set and a systematic stable set respectively, if they are stable sets.

In the case where $2/3 < v(2) \leq 3/4$, if we can find a stable set K' in the small triangle GHI, then as is easily checked $K' \cup K_{e.d.}$ is a

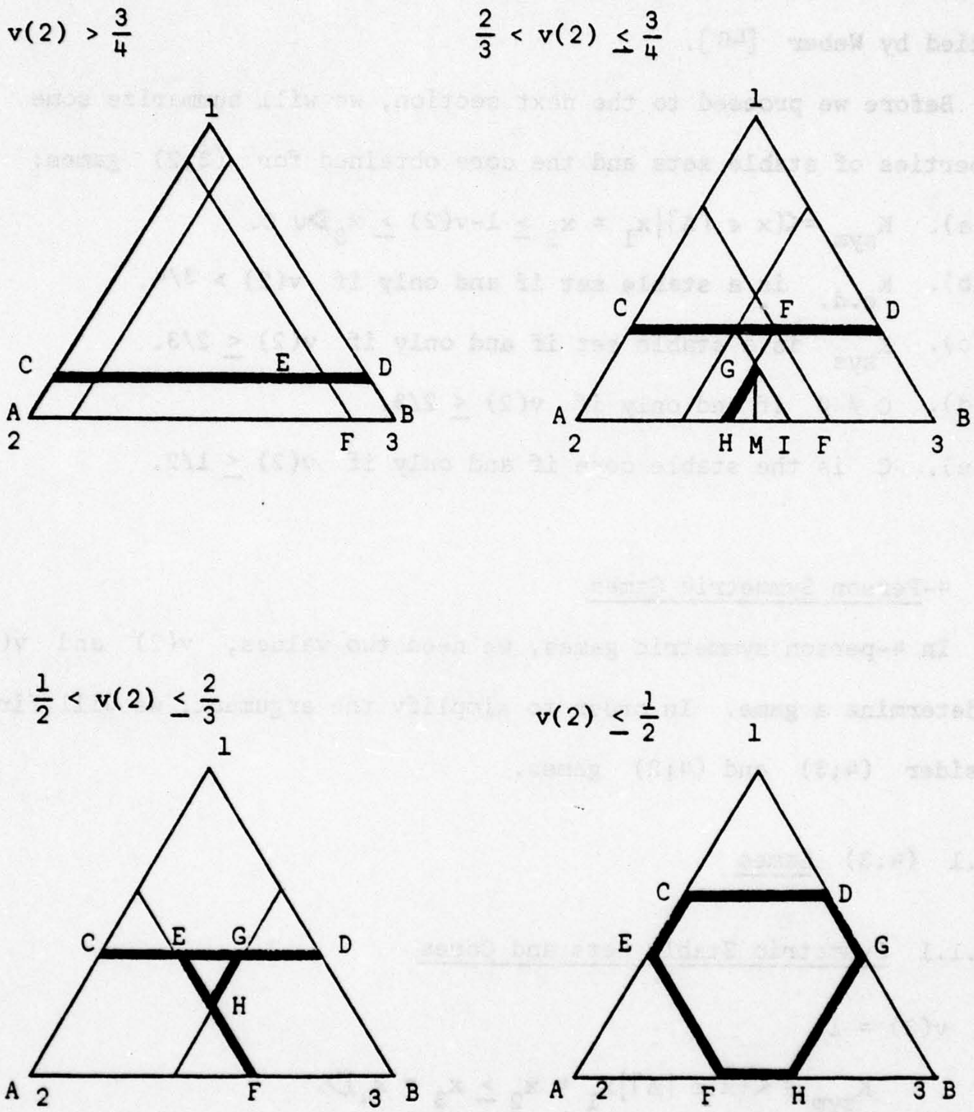


Figure 2.2 Stable set obtained systematically for (3;2)

stable set. For example, we can take the middle line GM as K'. The generalization of this type of stable set for (n;n-1) games have been studied by Weber [40].

Before we proceed to the next section, we will summarize some properties of stable sets and the core obtained for (3;2) games:

(a). $K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq 1-v(2) \geq x_3\}\rangle \cup C$.

(b). $K_{e.d.}$ is a stable set if and only if $v(2) > 3/4$.

(c). K_{sys} is a stable set if and only if $v(2) \leq 2/3$.

(d). $C \neq \emptyset$ if and only if $v(2) \leq 2/3$.

(e). C is the stable core if and only if $v(2) \leq 1/2$.

2.2 4-Person Symmetric Games

In 4-*person* symmetric games, we need two values, $v(2)$ and $v(3)$, to determine a game. In order to simplify the argument, we will first consider (4;3) and (4;2) games.

2.2.1 (4:3) Games

2.2.1.1 Symmetric Stable Sets and Cores

$v(3) = 1$:

$$K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq x_3 = x_4\}\rangle.$$

$$C = \emptyset.$$

$3/4 < v(3) < 1$:

$$K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq x_3 = x_4 \geq 1-v(3)\}\rangle$$
$$\cup \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq 1-v(3) \geq x_3 \geq x_4\}\rangle.$$

$$C = \emptyset.$$

$v(3) = 3/4$:

$$K_{sym} = \langle \{x \in \lceil A \rceil | x_1 = x_2 \geq 1/4 \geq x_3 \geq x_4\} \rangle \cup C$$

where $C = \langle \{x \in \lceil A \rceil | x_1 = x_2 = x_3 = x_4 = 1/4\} \rangle$.

$1/2 < v(3) < 3/4$:

$$K_{sym} = \langle \{x \in \lceil A \rceil | x_1 = x_2 \geq 1\text{-}v(3) \geq x_3 \geq x_4\} \rangle \cup C$$

where $C = \langle \{x \in \lceil A \rceil | x_1 \leq 1\text{-}v(3)\} \rangle$.

$v(3) \leq 1/2$:

$$K_{sym} = C$$

where $C = \langle \{x \in \lceil A \rceil | x_1 \leq 1\text{-}v(3)\} \rangle$.

These cases are illustrated in Figure 2.3.

2.2.1.2 Stable Sets Obtained Systematically

An analogue of the systematic way for (3;2) games is given as follows.

Systematic way (for (4;3) games):

Define

$$A_1 = \{x \in A | \text{there is no } y \in A \text{ such that } y \text{ dom } x \text{ via } \{2,3,4\}\},$$
$$A_2 = \{x \in A_1 | \text{there is no } y \in A_1 \text{ such that } y \text{ dom } x \text{ via } \{1,3,4\}\},$$
$$A_3 = \{x \in A_2 | \text{there is no } y \in A_2 \text{ such that } y \text{ dom } x \text{ via } \{1,2,4\}\}$$

and

$$A_4 = \{x \in A_3 | \text{there is no } y \in A_3 \text{ such that } y \text{ dom } x \text{ via } \{1,2,3\}\}.$$

If A_4 is a stable set, then $K = A_4$.

Similarly as before, we define

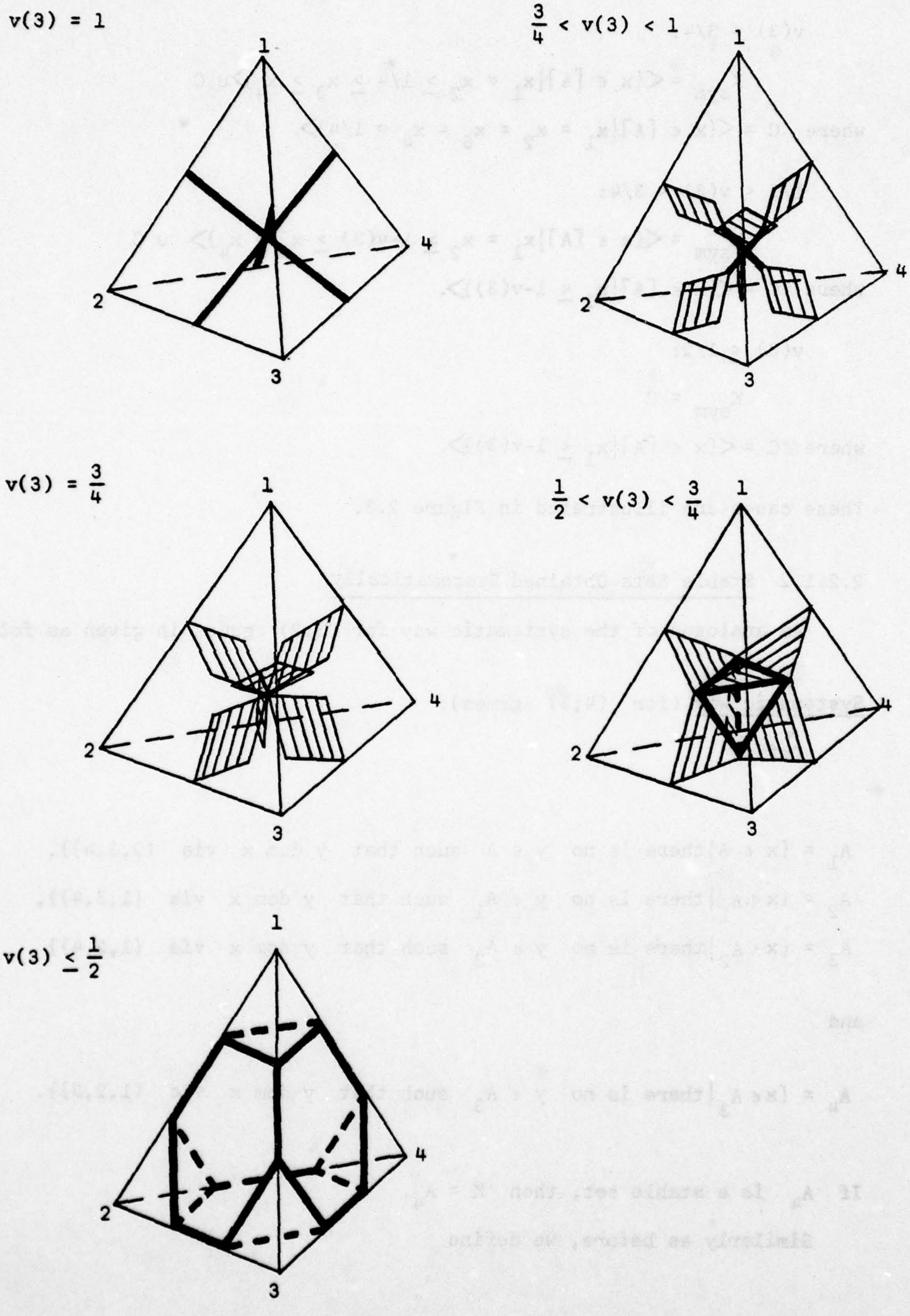


Figure 2.3 Symmetric stable sets for (4;3)

$A'_1 = \{x \in A | x_1 = 1-v(3)\}$,

$A'_2 = \{x \in A | x_1 < 1-v(3),\ x_2 = 1-v(3)\}$,

$A'_3 = \{x \in A | x_1 < 1-v(3),\ x_2 < 1-v(3),\ x_3 = 1-v(3)\}$,

$A'_4 = \{x \in A | x_1 < 1-v(3),\ x_2 < 1-v(3),\ x_3 < 1-v(3),\ x_4 = 1-v(3)\}$

$A'_5 = \{x \in A | x_1 < 1-v(3),\ x_2 < 1-v(3),\ x_3 < 1-v(3),\ x_4 < 1-v(3)\}$

and

$$K_{e.d.} = A'_1 \text{ and } K_{sys} = \bigcup_{i=1}^{5} A'_i.$$

Then the above systematic way reaches $K_{e.d.}$ if $v(3) > 5/6$ and K_{sys} if $v(3) \leq 2/3$. These cases are illustrated in Figure 2.4. In the case where $2/3 < v(3) \leq 5/6$, although we will not write it down, it is possible to find stable sets in the same way as we did in the case where $2/3 < v(2) \leq 3/4$ for (3;2) games.

Finally we will summarize the results obtained for (4;3) games.

(a). $K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq x_3 = x_4 \geq 1-v(3)\}\rangle$

$\cup \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq 1-v(3) \geq x_3 \geq x_4\}\rangle \cup C.$

(b). $K_{e.d.}$ is a stable set if and only if $v(3) > 5/6$.

(c). K_{sys} is a stable set if and only if $v(3) \leq 2/3$.

(d). $C \neq \emptyset$ if and only if $v(3) \leq 3/4$.

(e). C is the stable core if and only if $v(3) \leq 1/2$.

2.2.2 (4;2) Games

2.2.2.1 Symmetric Stable Sets and Cores

$2/3 \leq v(2)$:

$K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 = x_3 = 1/3,\ x_4 = 0\}\rangle$

$C = \emptyset$.

$v(3) > \frac{5}{6}$

$v(3) \leq 2/3$

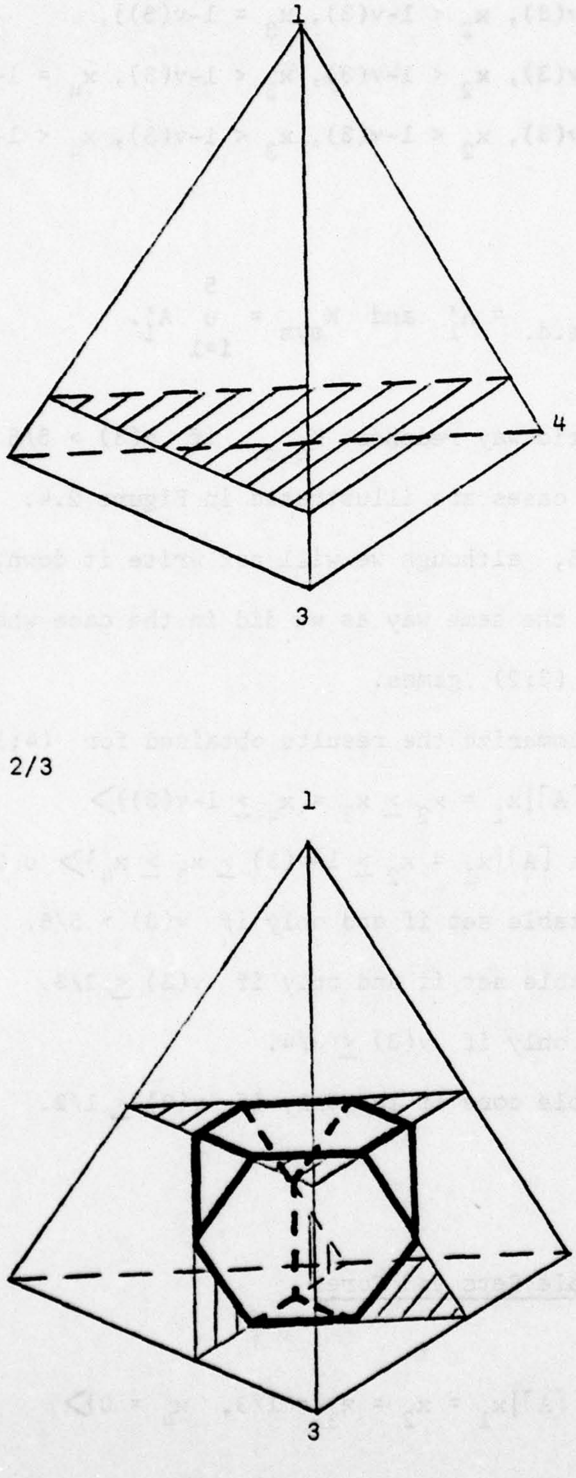


Figure 2.4 Stable sets obtained systematically for (4;3)

$1/2 < v(2) < 2/3$:

$$K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 = x_3 \geq v(2)/2 \geq x_4\}\rangle$$

$$C = \emptyset.$$

$v(2) = 1/2$:

$$K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 = x_3 \geq 1/4 (= v(2)/2 = (1-v(2))/2) \geq x_4\}\rangle \cup C$$

where $C = \langle\{x \in \lceil A \rceil | x_1 = x_2 = x_3 = x_4 = 1/4\}\rangle$.

$1/3 < v(2) < 1/2$:

$$K_{sym} = \langle\{x \in \lceil A \rceil | x_1 = x_2 = x_3 \geq (1-v(2))/2 \geq x_4, \quad x_1 + x_4 < v(2)\}\rangle \cup C$$

where $C = \langle\{x \in \lceil A \rceil | x_1 + x_2 \leq 1-v(2)\}\rangle$.

$v(2) \leq 1/3$:

$$K_{sym} = C$$

where $C = \langle\{x \in \lceil A \rceil | x_1 + x_2 \leq 1-v(2)\}\rangle$.

These cases are illustrated in Figure 2.5.

2.2.2.2 Stable Sets Obtained Systematically

It is easily known that the analogue of the above systematic way for (4;3) games does not work well for (4;2) games. So we will propose the following scheme instead.

Systematic way (for (4;2) games):

Let $\omega = \max(v(2)/2, (1-v(2))/2)$. Define

$$A_1 = \{x \in A | x_1 = x_2 = \omega, \quad x_3 \geq \omega, \quad x_4 \leq \omega\},$$
$$A_2 = \{x \in A | x_1 = x_2 = \omega, \quad x_3 \leq \omega, \quad x_4 \geq \omega\},$$
$$A_3 = \{x \in A | x_1 = \omega, \quad x_2 \leq \omega, \quad x_3 = \omega, \quad x_4 \geq \omega\},$$

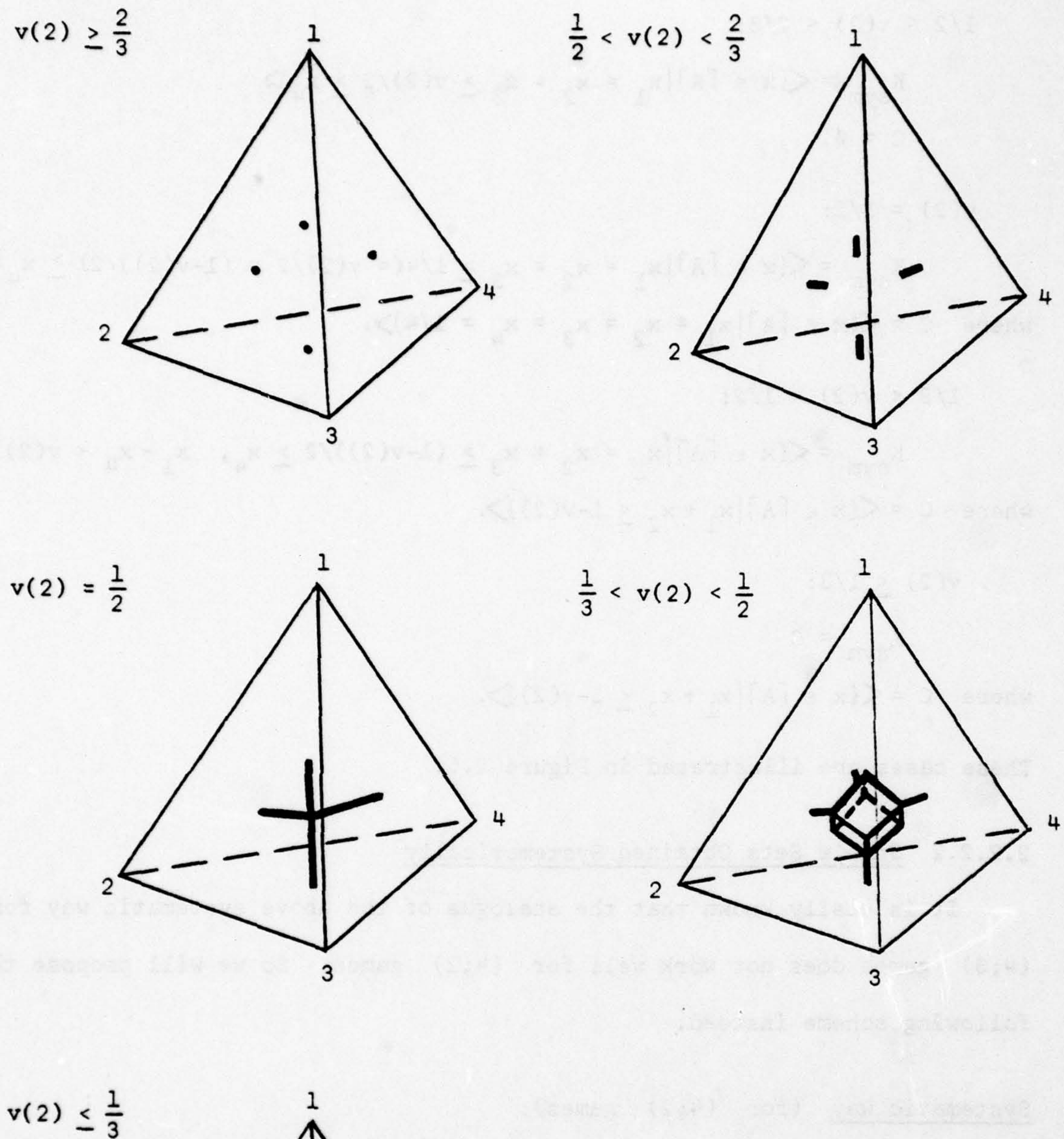


Figure 2.5 Symmetric stable sets for $(4;2)$

and

$$A_4 = \{x \in A | x_1 \leq \omega, \ x_2 = \omega, \ x_3 = \omega, \ x_4 \geq \omega\}.$$

Finally let $K = (\bigcup_{i=1}^{4} A_i) \cup C$.

If $v(2) \leq 2/3$, then this systematic way works well and we get the following stable set K:

$1/2 < v(2) \leq 2/3$:

K consists of four lines AE, BF, CG, DH.

$1/3 < v(2) \leq 1/2$:

K consists of four lines AE, BF, CG, DH and the core.

$v(2) \leq 1/3$:

K is the core.

These cases are illustrated in Figure 2.6.

We will call this K the systematic stable set for $(4;2)$ games and denote it by K_{sys}.

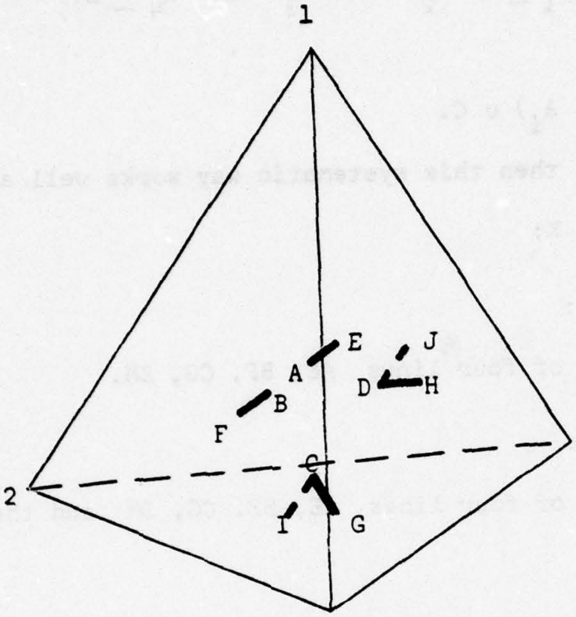
2.2.2.3 Semi-symmetric Stable Sets

Now if we replace the four lines AE, BF, CG, DH in the above K_{sys} by the four lines AE, BF, CI, DJ, then another type of stable sets will be obtained, namely:

$1/2 < v(2) \leq 2/3$:

$$K = \{x \in A | x_1 = x_2 = v(2)/2, \ x_3 \text{ or } x_4 \geq v(2)/2\}$$
$$\cup \{x \in A | x_1 \text{ or } x_2 \geq v(2)/2, \ x_3 = x_4 = v(2)/2\}.$$

$\frac{1}{2} < v(2) \leq \frac{2}{3}$



$\frac{1}{3} < v(2) \leq \frac{2}{3}$

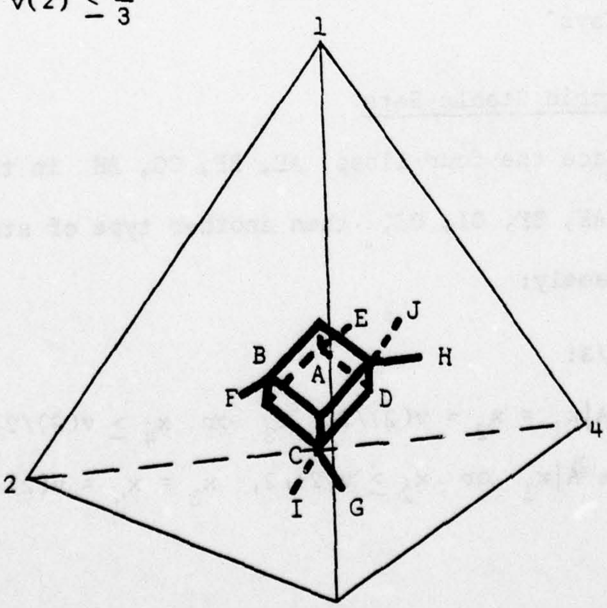


Figure 2.6 Semi-symmetric stable sets for (4;2)

$1/3 < v(2) \leq 1/2$:

$$K = \{x \in A \mid x_1 = x_2 = (1-v(2))/2, \ \ x_3 \ \text{or} \ \ x_4 \geq (1-v(2))/2\}$$
$$\cup \ \{x \in A \mid x_1 \ \text{or} \ \ x_2 \geq (1-v(2))/2, \ \ x_3 = x_4 = (1-v(2))/2\} \cup C$$

where $C = \langle\{x \in \lceil A \rceil \mid x_1 + x_2 \leq 1-v(2)\}\rangle$.

$v(2) \leq 1/3$:

$$K = C$$

where $C = \langle\{x \in \lceil A \rceil \mid x_1 + x_2 \leq 1-v(2)\}\rangle$.

This stable set can be condensed into the following one expression:

$$K = \{x \in A \mid x_1 = x_2 = \omega/2, \ \ x_3 \ \text{or} \ \ x_4 \geq \omega/2\}$$
$$\cup \ \{x \in A \mid x_1 \ \text{or} \ \ x_2 \geq \omega/2, \ \ x_3 = x_4 = \omega/2\} \ \cup C$$

where $\omega = \max(v(2), 1-v(2))$ *and* $v(2) \leq 2/3$.

Now this K is considered to be semi-symmetric in the sense that it is unchanged even if we exchange the coalition $\{1,2\}$ with the coalition $\{3,4\}$ and permute the players within $\{1,2\}$ and $\{3,4\}$. Thus we call this K the semi-symmetric stable set and denote it by $K_{s,s}$.

The results obtained for $(4;2)$ games are summarized as follows:

(a). If $v(2) \geq 2/3$, then there is a finite symmetric stable set $K_{sym} = \langle\{x \in \lceil A \rceil \mid x_1 = x_2 = x_3 = 1/3, \ \ x_4 = 0\}\rangle$.

(b). If $v(2) \leq 2/3$, then a symmetric stable set is given by

$$K_{sym} = \langle\{x \in \lceil A \rceil \mid x_1 = x_2 = x_3 \geq \max(v(2)/2, \ (1-v(2))/2) \geq x_4, \ \ x_3 + x_4 < v(2)\}\rangle \cup \ C.$$

(c). If $v(2) \leq 2/3$, then K_{sys} and $K_{s,s}$ are stable sets.

(d). $C \neq \emptyset$ if and only if $v(2) \leq 1/2$.

(e). C is the stable core if and only if $v(2) \leq 1/3$.

2.2.3 General 4-Person Symmetric Games

Symmetric stable sets for general 4-person symmetric games have been obtained by Nering [24]. Since these stable sets are rather complicated, we will not describe them. However, we will point out the following properties of the core:

(a). $C \neq \emptyset$ if and only if $v(2) \leq 1/2$ and $v(3) \leq 3/4$.

(b). C is the stable core if and only if $v(2) \leq 1/2$, $v(3) \leq 3/4$ and $v(4) + v(2) \geq 2v(3)$.

2.3 General 4-Person Games

The fact that every 4-person game (not necessarily symmetric) has at least one stable set has recently been announced (private communication) by O.N. Bondareva, T.E. Kulakovskaja and N.I. Naumova in Leningrad.

CHAPTER III

SURVEY OF SOME RESULTS

This chapter will be devoted to a survey of some known results related to stable sets and cores for symmetric games.

3.1 $(n;k)_b$ Games with $n/2 < k < n$

First let us consider $(n;k)_b$ games with $n/2 < k < n$. Recall $(n;k)_b$ games are given by

$$v(s) = \begin{cases} 1 & \text{for } s \geq k \\ 0 & \text{for } s < k. \end{cases}$$

This means that a coalition of k or more players can obtain the maximum possible amount and a smaller coalition is totally defeated. The symmetric stable sets for $(n;k)_b$ games with $n/2 < k < n$ were fully analyzed by Bott [3].

Theorem 3.1 (Bott): Let $p = n-k+1$ and write $n = sp+r$ where $0 \leq r \leq p$. Let

$$K_{sym} = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_p \geq x_{p+1} = \ldots = x_{2p}$$

$$\geq \ldots \geq x_{(s-1)p+1} = \ldots = x_{sp} \geq x_{sp+1} = \ldots = x_{sp+r} = 0\} \rangle .$$

Then K_{sym} is the unique symmetric stable set.

This theorem implies that all members of a particular blocking (or veto power) coalition (namely the smallest coalition which is large enough to

stop its complement from winning) will receive the same amount in an imputation of the symmetric stable set. Furthermore the defeated r players are completely exploited.

In the <u>zero-sum</u> case (namely n is odd and $k = (n+1)/2$) the symmetric stable set consists of a finite set of imputations, formed by permuting the coordinates of the imputation

$$(\underbrace{2/(n+1),\ldots,2/(n+1)}_{n-k+1},\underbrace{0,\ldots,0}_{k-1}) .$$

This is the <u>main</u> <u>simple</u> stable set of the <u>simple</u> <u>majority</u> game.

Finally we point out that if $k \leq n/2$, then there is a finite symmetric stable set consisting of all the permutations of the coordinates of the imputation

$$(\underbrace{1/(n-k+1),\ldots,1/(n-k+1)}_{n-k+1},\underbrace{0,\ldots,0}_{k-1}) .$$

Note that such games are not <u>superadditive</u>, i.e, they do not have the property that

$$v(S \cup T) \geq v(S) + v(T) \quad \text{whenever} \quad S \cap T = \emptyset.$$

3.2 (n;n-1) <u>Games</u>

(n;n-1) games are given by

$$0 < v(n-1) \leq 1$$

and

$$v(s) = 0 \quad \text{for all} \quad s < n-1.$$

This means that only coalitions with 1, n-1 and n players enter into the problem and all coalitions with less than n-1 players are totally defeated.

A symmetric stable set and the extremely discriminatory stable set have been studied by Lucas [15] and Owen [27], respectively, as indicated by the following.

<u>Theorem 3.2</u>. (Lucas): Let

$$K_{sym} = \bigcup_{r=0}^{[[n/2]]} \langle \{x \in \lceil A \rceil \mid x_1 = x_2 \geq \ldots \geq x_{2r-1} = x_{2r} \geq 1-v(n-1)$$
$$\geq x_{2r+1} \geq \ldots \geq x_n\} \rangle$$

where $[[n/2]]$ is the greatest integer in $n/2$.

Then K_{sym} is a symmetric stable set.

<u>Theorem 3.3</u>. (Owen): Let

$$K_{e.d.} = \{x \in A \mid x_1 = 1-v(n-1)\}.$$

Then $K_{e.d.}$ is a stable set if and only if $v(n-1) > (2n-3)/(2n-2)$.

3.3 The Condition for a Nonempty Core

In the previous chapter, we obtained the following conditions for a nonempty core:

For (3;2) games, $v(2) \leq 2/3$.

For 4-person symmetric games, $v(2) \leq 1/2$ and $v(3) \leq 3/4$.

The following well known theorem gives us a generalization of these conditions.

Theorem 3.4: n-Person symmetric games have a non-empty core if and only if $v(s) \leq s/n$ for all $s < n$.

The geometric interpretation of this theorem is that if we plot the $n+1$ points $(s,v(s))$ $(s = 0,1,\ldots,n)$ in the plane as in Figure 3.1, then the core is nonempty if and only if the point $(n,v(n))$ is "visible" from the origin $(0,v(0))$, i.e., the other points fall below the line through these two points.

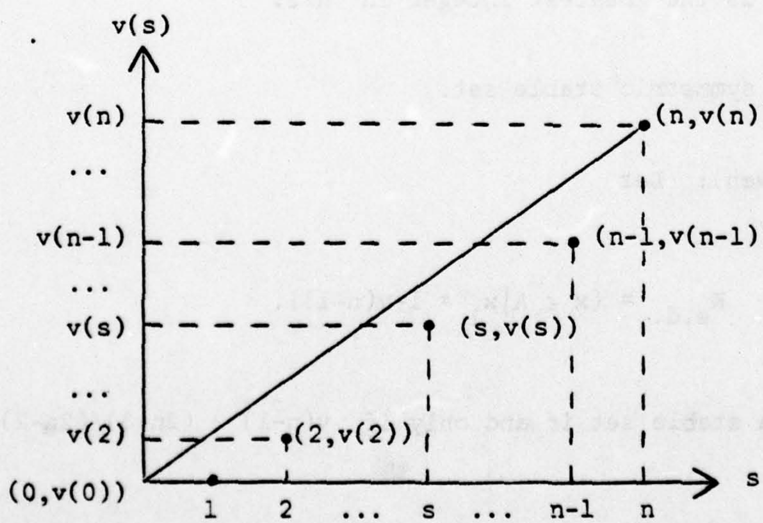


Figure 3.1 The condition for a nonempty core

3.4 The Condition for a Stable Core

As the final result of this chapter, we will point out the following interesting statement for the existence of a stable core which is due to Shapley [35].

Theorem 3.5. (Shapley): Whenever $C \neq \emptyset$ in an n-person symmetric game, then C is a stable set if and only if

$$\frac{v(n)-\bar{v}(k)}{n-k} \geq \frac{v(t)-\bar{v}(k)}{t-k} \quad \text{for all } t,k \text{ with } 0 \leq k < t < n$$

where $\bar{v}$ denotes the cover of v. Namely $\bar{v}(k) = \max_{0 \leq s \leq k} v(s) \cdot (k/s)$.

CHAPTER IV

SYSTEMATIC AND SEMI-SYMMETRIC STABLE SETS

4.1 Systematic Stable Sets

We will start this section with the explicit definition of systematic sets for $(n;k)$ games.

Definition 4.1: A set $K \subseteq A$ is said to be the systematic set for $(n;k)$ games if

$$K = \bigcup_{i(1)<\dots<i(n-k)} K_{i(1),\dots,i(n-k)}$$

where

$$K_{i(1),\dots,i(n-k)} = \{x \in A \mid x_{i(1)} = \dots = x_{i(n-k)} = (1-v(k))/(n-k);$$
$$x_i < (1-v(k))/(n-k) \text{ for all } i \leq (n-k),\ i \neq i(1),\dots,i(n-k);$$
$$x_i > (1-v(k))/(n-k) \text{ for at least one } i>i(n-k) \text{ if } i(n-k)<n\}.$$

We will denote this set K by $K_{sys}(n;k)$. Obviously $K_{sys}(n;k) \cap C \neq \emptyset$. As a generalization of the results obtained in Chapter II, we have the following theorem.

Theorem 4.1: $K_{sys}(n;k) \cup C$ is a stable set for $(n;k)$ games if and only if $v(k) \leq 2/(n-k+2)$.

Proof: Sufficiency: Internal stability: First we notice the following simple but important fact.

Claim: If $x \in K_{i(1),\ldots,i(n-k)}$, then there is only one i^* with $x_{i^*} > (1-v(k))/(n-k)$ and $x_i < (1-v(k))/(n-k)$ for all $i > i(n-k)$, $i \neq i^*$.

Proof of Claim: Suppose that the claim is false. Then

$$\sum_{i=1}^{n} x_i > (n-k+2)(1-v(k))/(n-k) \geq 1$$

since $v(k) \leq 2/(n-k+2)$. This is contrary to $x \in A$. □

Now pick any two elements, say x and y, in $K_{sys}(n;k) \cup C$ and assume x dom y via S with $|S| = k$.

Case (i) $x,y \in K_{sys}(n;k)$: Assume $x \in K_{i(1),\ldots,i(n-k)}$ and $y \in K_{j(1),\ldots,j(n-k)}$. Let x_{i^*} and y_{i^*} be greater than $(1-v(k))/(n-k)$. If $i(\ell) = j(\ell)$ for all $\ell = 1,\ldots,n-k$ then clearly x d$\not{o}$m y. Thus we assume $i(\ell) \neq j(\ell)$ for some $\ell = 1,\ldots,n-k$. Then there must exist at least $(n-k)$ i's such that $x_i \leq y_i$, $x_i \leq (1-v(k))/(n-k)$ for all i and $x_i < (1-v(k))/(n-k)$ for some i because of the claim shown above. Thus x d$\not{o}$m y.

Case (ii) $x \in C$, $y \in K_{sys}(n;k)$: Assume $y \in K_{j(1),\ldots,j(n-k)}$, then from the definition of $K_{j(1),\ldots,j(n-k)}$ there is at least one $j \in S$ with $y_j \geq (1-v(k))/(n-k)$ and thus $x_j > (1-v(k))/(n-k)$ for at least one $j \in S$. Since $x \in C$ and S is effective for x, $x_i \geq x_j$ for all $i \in S$ and $j \notin S$. Hence $x_i > (1-v(k))/(n-k)$ for all $i \notin S$. Therefore we obtain the contradiction

$$\sum_{i=1}^{n} x_i = \sum_{i \in S} x_i + \sum_{i \notin S} x_i > v(k) + (n-k)(1-v(k))/(n-k) = 1.$$

<u>External stability</u>: Take any $x \in A - (K_{sys}(n;k) \cup C)$. Let x' be the imputation obtained from x by permuting the coordinates into nonincreasing order. Since $x \notin C$, $\sum_{i=n-k+1}^{n} x'_i < v(k)$. If $x'_{n-k+1} < (1-v(k))/(n-k)$, then $x \in \text{Dom } C$. In fact, define y by

$$y_i = \begin{cases} (1-v(k))/(n-k) & \text{for } i = 1,\dots,n-k \\ x'_i + \varepsilon_i & \text{for } i = n-k+1,\dots,n \end{cases}$$

where

$$\sum_{i=n-k+1}^{n} \varepsilon_i = v(k) - \sum_{i=n-k+1}^{n} x'_i, \quad \varepsilon_i > 0 \text{ for all } i = n-k+1,\dots,n$$

and

$$y_i \le (1-v(k))/(n-k) \quad \text{for all } i = n-k+1,\dots,n.$$

Then $y \in C$ and y dom x' via $\{n-k+1,\dots,n\}$. Thus we have $x'_1 > (1-v(k))/(n-k)$, $x'_{n-k+1} \ge (1-v(k))/(n-k)$ and $x'_{n-k+2} < (1-v(k))/(n-k)$ since $x \notin C$ and $v(k) \le 2/(n-k+2)$.

Now we assume that $x_{i(1)},\dots,x_{i(n-k)} \ge (1-v(k))/(n-k)$ and $i(1) < \dots < i(n-k)$. Define y by

$$y_i = \begin{cases} (1-v(k))/(n-k) & \text{for all } i = i(1),\dots,i(n-k) \\ x_i + \varepsilon_i & \text{for } i \ne i(1),\dots,i(n-k) \end{cases}$$

where

$$\sum_{i \neq i(1),\ldots,i(n-k)} \varepsilon_i = \sum_{\ell=1}^{n-k} x_{i(\ell)} - (1-v(k)), \quad \varepsilon_i > 0$$

for all $i \neq i(1),\ldots,i(n-k)$

and

$$y_i < (1-v(k))/(n-k) \quad \text{for all} \quad i \neq i(1),\ldots,i(n-k).$$

Then $y \in K_{i(1),\ldots,i(n-k)}$ and y dom x via $N - \{i(1),\ldots,i(n-k)\}$.

Necessity: Assume $v(k) > 2/(n-k+2)$. Then $K_{sys}(n;k)$ does not satisfy internal stability since there may exist two or more i's with $x_i > (1-v(k))/(n-k)$. □

Corollary 4.1: For $(n;2)$ games, $K_{sys}(n;2) \cup C$ is a stable set if and only if C is not empty.

Proof: For $(n;2)$ games, $C \neq \emptyset$ if and only if $v(2) \leq 2/n$. □

Now under the same condition in Theorem 4.1, a symmetric stable set is easily obtained.

Theorem 4.2: Let

$$K_{sym} = \langle\{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-k+1} \geq (1-v(k))/(n-k) \geq x_{n-k+2} \geq \ldots \geq x_n\}\rangle \cup C.$$

Then K_{sym} is a symmetric stable set for $(n;k)$ games if and only if $v(k) \leq 2/(n-k+2)$.

Proof: Sufficiency: Internal stability: Pick any x,y in K_{sym} and assume x dom y via $S_x|\{n-k+1,\ldots,n\}_y$ where $|S_x| = k$.

Case (i) $x,y \in K_{sym}-C$: First we note that S_x should be $(\{i(0)\} \cup \{n-k+2,\ldots,n\})_x$ where $i(0) \in \{1,\ldots,n-k+1\}_x$. In fact, if S_x contains two or more, say ℓ, elements from $\{1,\ldots,n-k+1\}_x$, then we get the contradiction

$$\sum_{i=1}^{n} x_i > \ell \cdot ((1-v(k))/(n-k)) + 1-v(k) \geq ((n-k+2)/(n-k)) \cdot (1-v(k)) \geq 1$$

since $v(k) \leq 2/(n-k+2)$. Therefore we obtain $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$.

Case (ii) $x \in C$, $y \in K_{sym}-C$: Without loss of generality, we assume $S_x = \{n-k+1,\ldots,n\}_x$. Then $x_{n-k+1} > (1-v(k))/(n-k)$ since $y_{n-k+1} \geq (1-v(k))/(n-k)$. Hence we have the contradiction

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-k} x_i + \sum_{i=n-k+1}^{n} x_i > (n-k)(1-v(k))/(n-k) + v(k) = 1.$$

External stability: Take any $x \in \lceil A \rceil - K_{sym}$. Then as in Theorem 4.1, we obtain $x_1 > (1-v(k))/(n-k)$, $x_{n-k+1} \geq (1-v(k))/(n-k)$ and $x_{n-k+2} < (1-v(k))/(n-k)$. Since $x \notin K_{sym}$, there is at least one $i^* \in \{1,\ldots,n-k\}$ with $x_{i^*} > x_{i^*+1}$. Define y by

$$y_i = \begin{cases} x_{n-k+1} + \varepsilon_i & \text{for } i = 1,\ldots,n-k+1 \\ x_i + \varepsilon_i & \text{for } i = n-k+2,\ldots,n \end{cases}$$

where

$$\sum_{i=1}^{n} \varepsilon_i = \sum_{i=1}^{n-k+1} x_k - (n-k+1)x_{n-k+1}, \quad \varepsilon_i > 0 \text{ for all } i = 1,\ldots,n$$

and

$$y_i < (1-v(k))/(n-k) \quad \text{for all } i = n-k+2,\ldots,n.$$

Then $y \in K_{sym}$ and y dom x via $\{n-k+1,\ldots,n\}$.

Necessity: If we assume $v(k) > 2/(n-k+2)$, then clearly internal stability is not satisfied. □

Corollary 4.2: For $(n;2)$ games, K_{sym} is a stable set if and only if C is not empty.

Proof: This proof is the same as that of Corollary 4.1. □

4.2 Semi-symmetric Stable Sets

Before going into symmetric games, let us consider our problem in a more general setup.

4.2.1 Generalized k-Quota Stable Sets

Throughout this section, we will consider (N,k) games. Recall $(N;k)$ games are given by

$$v(S) \begin{cases} > 0 & \text{for all } S \text{ with } |S| = k \\ = 0 & \text{for all } S \text{ with } |S| < k. \end{cases}$$

Definition 4.2: An n-dimensional vector ω is said to be a semi-quota for (N,k) games if

$$\sum_{i\in S} \omega_i = v(S) \text{ for all } S \text{ with } |S| = k \text{ and } \omega_i \geq 0.$$

Let $\Omega = \sum_{i\in N} \omega_i - 1.$

Definition 4.3: Consider (N,k) games with $n = qk + r$ $(q \geq 2,$ $0 \leq r \leq k-1)$. Let $\{S_1,\dots,S_{q+1}\}$ be a partition of N such that $|S_j| = k$ for all $j = 1,\dots,q$ and $|S_{q+1}| = r$. If $S_{q+1} \neq \emptyset$, then let $S'_{q+1} = S_{q+1} \cup T$ where $|S'_{q+1}| = k$ and $T = \bigcup_{j=1}^{q} T_j$ $(T_j \subseteq S_j)$. A set $K \subseteq A$ is said to be a semi-symmetric set for (N,k) games with $n = qk + r$ $(q \geq 2)$ with respect to ω if

$$K = \bigcup_{j=1}^{q+1} K_j$$

where

$$K_j = \{x \in A \mid \sum_{i\in S_j} x_i = \sum_{i\in S_j} \omega_i - \Omega;\ x_i \geq \omega_i \text{ for at least one } i \in S_j;$$

$$x_i = \omega_i \text{ for all } i \notin S_j\} \text{ for } j = 1,\dots,q$$

and

$$K_{q+1} = \begin{cases} \emptyset & \text{if } S_{q+1} = \emptyset \\ \{x \in A \mid \sum_{i\in S'_{q+1}} x_i = \sum_{i\in S'_{q+1}} \omega_i - \Omega;\ x_i \geq \omega_i \text{ for all } i \in T; & \\ x_i = \omega_i \text{ for all } i \notin S'_{q+1}\} & \text{if } S_{q+1} \neq \emptyset \end{cases}$$

The symbol $K_{s,s,\omega}(N,k)$ will be used to denote this set K.

Definition 4.4: Consider (N,k) games with $n = 2k-1$. Let $\{S_1, S_2, i(0)\}$ be a partition of N with $|S_1| = |S_2| = k-1$. A set $K \subseteq A$ is said to be a semi-symmetric set for (N,k) games with $n = 2k-1$ with respect to ω if

$$K = K_1 \cup K_2$$

where

$$K_1 = \{x \in A \mid \sum_{i \in S_1 \cup \{i(0)\}} x_i = \sum_{i \in S_1 \cup \{i(0)\}} \omega_i - \Omega;\ x_i \geq \omega_i \text{ for at least one } i \in S_1 \cup \{i(0)\};\ x_i = \omega_i \text{ for all } i \in S_2\}$$

and

$$K_2 = \{x \in A \mid \sum_{i \in S_2 \cup \{i(0)\}} x_i = \sum_{i \in S_2 \cup \{i(0)\}} \omega_i - \Omega;\ x_{i(0)} \geq \omega_{i(0)};\ x_i = \omega_i \text{ for all } i \in S_1\}.$$

We will denote this set K by $K_{s,s,\omega}(N,(n+1)/2)$. These definitions enable us to state and prove the following theorems. The first one is due to Shapley [31] and Kalisch [14].

Theorem 4.3: Assume $k = 2$. (a). If $\Omega = 0$, then both $K_{s,s,\omega}(N,k)$ and $K_{s,s,\omega}(N,(n+1)/2)$ are stable sets. (b). If $\Omega > 0$ and every K_j in the above definitions is not empty, then both $K_{s,s,\omega}(N,k)$ and

$K_{s,s,\omega}$ $(N,(n+1)/2)$ are stable sets.

The following two theorems will give us a generalization of the first part of Theorem 4.3.

<u>Theorem 4.4</u>: Consider (N,k) games with $n = qk + r$ $(q \geq 2,\ 0 \leq r \leq k-1)$. Assume $\Omega = 0$ and $\sum_{i\in S} \omega_i \geq v(S)$ for all S with $k < |S| \leq 2k-2$. (a). If $S_{q+1} = \emptyset$, then $K_{s,s,\omega}$ (N,k) is a stable set. (b). Assume $S_{q+1} \neq \emptyset$. Then $K_{s,s,\omega}$ (N,k) is a stable set if and only if we can take T_j's so that $|T_j| \leq 1$ for all $j = 1,\ldots,q$ or $\omega_i = 0$ for all $i \in S_{q+1}$.

<u>Proof of (a)</u>: <u>Internal stability</u>: Since $\sum_{i\in S} \omega_i \geq v(S)$ for all S with $|S| \leq 2k-2$ and $v(S) = 0$ for all S with $|S| < k$, it is sufficient to consider dominations via sets having exactly k members.

Pick any two elements x,y in $K_{s,s,\omega}$ (N,k). Assume $x \in K_j$ and $y \in K_{j'}$. If $j = j'$, then x d$\emptyset$m y. Thus we assume $j \neq j'$ and x dom y via S. From the definition of K_j, $S \subset S_j \cup S_{j'}$, $S \cap S_j \neq \emptyset$ and $S \cap S_{j'} \neq \emptyset$. Furthermore $x_i > y_i = \omega_i$ for all $i \in S \cap S_j$ and $x_i = \omega_i$ for all $i \in S \cap S_{j'}$. Therefore we obtain $\sum_{i\in S} x_i > \sum_{i\in S} \omega_i = v(S)$ which contradicts the effectiveness of S.

<u>External stability</u>: Take any $x \in A - K_{s,s,\omega}$ (N,k) and let $S_- = \{i \in N | x_i < \omega_i\}$. If $|S_-| \geq k$, then x is dominated by ω. Thus we assume $|S_-| \leq k-1$. Let $|S_-| = m$ and i^* be one of the players with the maximum value of $x_i - \omega_i$, then $x_{i^*} > \omega_{i^*}$ since $x \notin K_{s,s,\omega}$ (N,k).

<u>Case (i)</u> $S_- \subseteq S_j$ for some $j = 1,\ldots,q$: Since $x \notin K_{s,s,\omega}$ (N,k),

there is some $i \notin S_j$ with $x_i > \omega_i$ and thus $\sum_{i \notin S_j} x_i > \sum_{i \notin S_j} \omega_i$ or $\sum_{i \in S_j} x_i < \sum_{i \in S_j} \omega_i = v(S_j)$. Hence we can take some $y \in K_j$ which dominates x via S_j.

<u>Case (ii)</u> $S_- \not\subseteq S_j$ for any $j = 1,\ldots,q$: For j with $S_j \cap S_- \neq \emptyset$, let $S_-^j = S_j \cap S_-$ and take $R_j \subseteq S_j - S_-^j$ satisfying $|R_j| = k-m$ and $i^* \in R_j$. For some j, if $\sum_{i \in R_j \cup S_-^j} x_i < \sum_{i \in R_j \cup S_-^j} \omega_i$ then we can take some $y \in K_j$ which dominates x via $R_j \cup S_-$. Thus we must have $\sum_{i \in R_j \cup S_-^j} x_i \geq \sum_{i \in R_j \cup S_-^j} \omega_i$ for all j with $S_-^j \neq \emptyset$. Since $x_i \geq \omega_i$ for all $i \notin \bigcup_{\{j|S_-^j \neq \emptyset\}} (R_j \cup S_-^j)$ and moreover $i^* \notin \bigcup_{\{j|S_-^j \neq \emptyset\}} (R_j \cup S_-^j)$, we have the contradiction $\sum_{i \in N} x_i > \sum_{i \in N} \omega_i = 1$. □

<u>Proof of (b)</u>: <u>Sufficiency</u>: <u>Internal stability</u>: This is proved in the same way as above.

<u>External stability</u>: It suffices to consider the case where $S_- \cap S_{q+1} \neq \emptyset$.

<u>Case (i)</u>: $S_- \subseteq S_{q+1}$: It is obvious that there is some $y \in K_{q+1}$ which dominates x via S'_{q+1}.

<u>Case (ii)</u> $S_- \not\subseteq S_{q+1}$: Let $S_-^{q+1} = S_- \cap S_{q+1}$. <u>(ii-I)</u> $i^* \in S_{q+1}$: For $j \neq q+1$ with $S_-^j \neq \emptyset$, take $R_j \subseteq S_j - S_-^j$ satisfying $|R| = k-m$ and $R_j \cap T_j = \emptyset$. For $q+1$, take $R_{q+1} \subseteq S'_{q+1} - S_-^{q+1}$ satisfying $|R_{q+1}| = k-m$ and $i^* \notin R_{q+1}$. <u>(ii-II)</u> $i^* \notin S_{q+1}$: Assume $i^* \in S_{j^*}$ $(j^* \neq q+1)$. For $j \neq j^*, q+1$, take R_j as above. For j^*, take $R_{j^*} \subseteq S_{j^*} - S_-^{j^*}$ satisfying $|R_{j^*}| = k-m$ and $i^* \notin R_{j^*}$. Finally, for $q+1$, take $R_{q+1} \subseteq S'_{q+1} - S_-^{q+1}$ satisfying $|R_{q+1}| = k-m$

and $R_{q+1} \cap T_{j*} = \emptyset$. Then in a manner similar to that in the proof of (a), we get a contradiction.

Necessity: Suppose $|T_{j(0)}| \geq 2$ for some $j(0) \in \{1,\ldots,q\}$ and $\omega_{i(0)} > 0$ for some $i(0) \in S_{q+1}$. Take $i', i'' \in T_{j(0)}$ and $i''' \in S_{j(0)}$ so that $\omega_i > 0$ for at least one of $i \in \{i',i'',i'''\}$. Let $i(+)$ be one of the players i', i'' and i''' with $\omega_{i(+)} > 0$ and let $\varepsilon = \min(\omega_{i(+)},\omega_{i(0)})$. Now define x by

$$x_i = \begin{cases} \omega_i & \text{for } i \neq i',i'',i''',i(0) \\ \omega_i - \varepsilon & \text{for } i = i(+),i(0) \\ \omega_i + \varepsilon & \text{for } i \in \{i',i'',i'''\} - \{i(+)\}. \end{cases}$$

Then $x \notin K_{s,s,\omega}(N,k)$. Moreover it is easily shown that $x \notin \mathrm{Dom}(\bigcup_{j \neq j(0),q+1} K_j)$. For all $S \subseteq S_{j(0)}$ with $|S| = k-1$ and $i(+) \in S$, $\sum_{i \in S} x_i = \sum_{i \in S} \omega_i$. Thus $x \notin \mathrm{Dom}\, K_{j(0)}$. Similary if $i(+) \notin T_{j(0)}$, then there is no $y \in K_{q+1}$ such that y dom x. If $i(+) \in T_{j(0)}$, there is also no $y \in K_{q+1}$ such that y dom x, since $\sum_{i \in S'_{q+1}-\{i(+)\}} x_i \geq \sum_{i \in S'_{q+1}-\{i(+)\}} \omega_i$. Therefore $K_{s,s,\omega}(N,k)$ does not satisfy external stability. □

Theorem 4.5: Consider (N,k) games with $n = 2k-1$. If $\Omega = 0$ and $\sum_{i \in S} \omega_i \geq v(S)$ for all S with $k \leq |S| \leq 2k-2$ then $K_{s,s,\omega}(N, (n+1)/2)$ is a stable set.

Proof: Internal stability: This is clear.

External stability: The same argument as in the proof of Theorem 4.4(a) holds except when $S_- \cap S_1 = \emptyset$, $S_- \cap S_2 \neq \emptyset$, $x_{i(0)} < \omega_{i(0)}$ and $i^* \in S_2$. In this case, the following conditions must be satisfied in order that X not be dominated by some $y \in K_{s,s,\omega}(N, (n+1)/2)$; $\sum_{i \in S_2} x_i \geq \sum_{i \in S_2} \omega_i$ and $\sum_{i \in R_1 \cup \{i(0)\}} x_i \geq \sum_{i \in R_1 \cup \{i(0)\}} \omega_i$ *for all* $R_1 \subseteq S_1$ with $|R_1| = k-m$. Now since $|R_1| = k-m \leq k-2$ and $S_- \cap S_1 = \emptyset$, there must exist some player $i \in S_1$ with $x_i = \omega_i$. Taking R_1 so that R_1 contains this i, we obtain another player $j \in S_1$ with $x_j = \omega_j$. Now again taking R_1 which contains i and j, we get player $k(\neq i,j)$ with $x_k = \omega_k$. Repeat this procedure. Then finally we get $x_i = \omega_i$ for all $i \in S_1$. This contradicts the condition $\sum_{i \in R_1 \cup \{i(0)\}} x_i \geq \sum_{i \in R_1 \cup \{i(0)\}} \omega_i$ for all $R_1 \subseteq S_1$ with $|R_1| = k-m$ since $x_{i(0)} < \omega_{i(0)}$. □

Remarks: (a). In Theorem 4.4(a), if $r+q \geq k$, then we can choose T_j's which satisfy the condition in this theorem. (b). In Theorems 4.4 and 4.5, if $k = 2$, then the conditions $|T_j| \leq 1$ for all $j = 1,\ldots,q$ and $\sum_{i \in S} \omega_i \geq v(S)$ for all S with $k \leq |S| \leq 2k-2$ are always satisfied. Thus the first part of Theorem 4.3 could be obtained as a corollary of Theorems 4.4 and 4.5.

Now let us return to symmetric games with the above three theorems in mind.

4.2.2 Symmetric Games

First we will explicitly define semi-symmetric sets for symmetric games.

Definition 4.3': Consider $(n;k)$ games with $n = qk + r$ $(q \geq 2,\ 0 \leq r \leq k-1)$. Let $\omega = \max(v(k)/k,\ (1-v(k))/(n-k))$ and $\Omega = n\omega - 1$. Define $\{S_1,\ldots,S_{q+1}\}$, S'_{q+1} and $\{T_1,\ldots,T_q\}$ as in Definition 4.3. Then the set K defined in Definition 4.3 is said to be the semi-symmetric set for $(n;k)$ games with $n = qk + r$ and is denoted by $K_{s,s}(n;k)$.

Definition 4.4': Consider $(n;k)$ games with $n = 2k-1$. Let $\omega = \max(v(k)/k,\ (1-v(k))/(n-k))$ and $\Omega = n\omega - 1$. Define S_1, S_2 and $\{i(0)\}$ as in Definition 4.4. Then the set K defined in Definition 4.4 is said to be the semi-symmetric set for $(n;k)$ games with $n = 2k-1$ and is denoted by $K_{s,s}(n;\ (n+1)/2)$.

For $(n;2)$ games, the next theorem will give us a more general result than that obtained by the direct application of Theorem 4.3.

Theorem 4.3': Consider $(n;2)$ games. Then $K_{s,s}(n;2) \cup C$ is a stable set if and only $v(2) \leq 2/(n-1)$.

Proof: For $K_{s,s}(3;2)$, this is trivial. (We note that $n = 3$ means $v(2) \leq 1$, namely, the condition is always satisfied.) Thus we will deal exclusively with $K_{s,s}(n;2) \cup C$ with $n \geq 4$ in what follows.

Necessity: This is clear. In fact, if $v(2) > 2/(n-1)$, then each K_j in Definition 4.3 is empty. Thus $K_{s,s}(n;2) \cup C$ is empty.

Sufficiency: If $v(2) \geq 2/n$, then this theorem follows from Theorem 4.3. Thus we assume $v(2) < 2/n$.

Internal stability: Pick any $x,y \in K_{s,s}(n;2) \cup C$.

Case (i) $x, y \in K_{s,s}(n;2) - C$: Assume $x \in K_j$, $y \in K_{j'}$ $(j \neq j')$ and x dom y via S. Then we get a contradiction since $((1-v(2))/(n-2) \cdot 2 > v(2)$.

Case (ii) $x \in C$, $y \in K_{s,s}(n;2) - C$: If x dom y, then we obtain the contradiction

$$\sum_{i=1}^{n} x_i > v(2) + ((1-v(2))/(n-2)) \cdot (n-2) = 1$$

since $y'_{n-1} = (1-v(2))/(n-2)$ where y' is the imputation obtained from y by permuting the coordinates into nonincreasing order.

External stability: Take any $x \in A-(K_{s,s}(n;2) \cup C)$. Then we must have $x'_1 > (1-v(2))/(n-2)$, $x'_{n-1} \geq (1-v(2))/(n-2)$ and $x'_n < (1-v(2))/(n-2)$ where x' is the imputation obtained from x by permuting the coordinates into nonincreasing order. Assume $x'_n = x_{i^*}$ and $i^* \in S_{j^*}$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i \in S_{j^*} \\ (1-v(2))/(n-2) & \text{for } i \notin S_{j^*} \end{cases}$$

where

$$2\varepsilon = \sum_{i \notin S_{j^*}} x_i - (1-v(2)) > 0.$$

Then $y \in K_{j^*}$ and y dom x via S_{j^*}. □

Remark: If $v(2) > 2/(n+1)$, then there is a finite symmetric stable set consisting of all imputations obtained by taking the permutations of the

indices of the imputation $(\underbrace{1/(n-1),\ldots,1/(n-1)}_{n-1}, 0)$.

We conclude this section by stating the counterparts of Theorems 4.4 and 4.5 without proof.

Theorem 4.4': Consider $(n;k)$ games with $n = qk + r$ $(q \geq 2,\ 0 \leq r \leq 1)$. Assume that the core consists only of one point. (a). If $S_{q+1} = \emptyset$, then $K_{s,s}(n;k)$ is a stable set. (b). Assume $S_{q+1} \neq \emptyset$. Then $K_{s,s}(n;k)$ is a stable set if and only if we can take T_j's satisfying $|T_j| \leq 1$ for $j = 1,\ldots,q$.

Theorem 4.5': Consider $(n;k)$ games with $n = 2k-1$. Assume that the core consists only of one point. Then $K_{s,s}(n; (n+1)/2)$ is a stable set.

4.3. Concluding Remarks

Although the results obtained in this chapter are somewhat limited, systematic and semi-symmetric type stable sets do merit further study in order to grasp the structure of stable sets for symmetric games. The following problems would be of particular interest.

(a). What are the systematic type stable sets if the condition in Theorem 4.1 is not satisfied? As shown in the proof, if this condition is violated, internal stability of $K_{sys}(n;k)$ does not hold. Thus some restrictions on $K_{sys}(n;k)$ would be required to maintain internal stability.

(b). Extensions of the second part of Theorem 4.3, along the same line as done for its first part, would be of interest.

(c). What are the semi-symmetric type stable sets for $(n;k)$ games with $n = qk + r$ and $r+q < k$? In this case, some kind of enlargement

of $K_{s,s}(n;k)$ would be required to preserve external stability.

(d). What are the semi-symmetric type stable sets for $(n;k)$ games with $k > [[(n+1)/2]]$?

(e). What is the relation between the systematic and the semi-symmetric stable sets?

CHAPTER V

SYMMETRIC STABLE SETS FOR (n;k) GAMES

In Chapters V and VI, we will be concerned with symmetric stable sets for symmetric games. Therefore we will assume, for simplicity of notation, that any imputation of A has its coordinates arranged into nonincreasing order unless we explicitly state otherwise.

Before stating the main results of this chapter, we will prove an important lemma which will be used frequently in the following.

<u>Lemma 5.1</u>: Let K be a symmetric stable set for $(n;k)$ games. Then if $x \in K-C$,

$$x_1 = x_2 = \dots = x_{n-k+1} > (1-v(k))/(n-k).$$

<u>Proof</u>: <u>Case (i)</u> $C \neq \emptyset$ (i.e., $v(k) > k/n$): we will first show that $x_1 = x_2 = \dots = x_{n-k+1}$. Suppose that $x_{i(0)} > x_{i(0)+1}$ for some $1 \leq i(0) \leq n-k$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i \neq i(0) \\ x_{i(0)} - (n-1)\varepsilon & \text{for } i = i(0) \end{cases}$$

where $0 < \varepsilon < (x_{i(0)} - x_{i(0)+1})/n$, i.e., $y_{i(0)} > y_{i(0)+1}$. Then $y_i > x_i$ for all $i = n-k+1,\dots,n$. Moreover $\{n-k+1,\dots,n\}_y$ is effective for y. In fact, if $\sum_{i=n-k+1}^{n} y_i > v(k)$, then $y_{n-k+1} > v(k)/k$. Thus we get the contradiction

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n-k} y_i + \sum_{i=n-k+1}^{n} y_i > v(k)\cdot(n-k)/k + v(k) = v(k)\cdot n/k > 1.$$

Therefore y dom x via $\{n-k+1,\ldots,n\}$. Hence $y \notin K$ and thus there must exist some $z \in K$ such that z dom y. Assume z dom y via $\{i(1),\ldots,i(k)\}_z | \{n-k+1,\ldots,n\}_y$. Then $z_{i(r)} > y_{n-k+r}$ for all $r = 1,\ldots,k$. However $y_{n-k+r} > x_{n-k+r}$ for all $r = 1,\ldots,k$. Thus we have $z_{i(r)} > x_{n-k+r}$ for all $r = 1,\ldots,k$ which means z dom x. This contradicts the fact that $x,z \in K$. Finally if we assume that $x_1 = \ldots = x_{n=k+1} \leq (1-v(k))/(n-k)$, then

$$\sum_{i=1}^{n} x_i \leq ((1-v(k))/(n-k)) \cdot n = 1 + (k-n \cdot v(k))/(n-k) < 1$$

which is contrary to $x \in A$.

Case (ii) $C \neq \emptyset$ (i.e., $v(k) \leq k/n$): Since $x \in K-C$, $x_{n-k+1} \geq (1-v(k))/(n-k)$ and $x_1 > (1-v(k))/(n-k)$. Suppose $x_{i(0)} > x_{i(0)+1}$ for some $1 \leq i(0) \leq n-k$ and define y as above. Then y dom x via $\{n-k+1,\ldots,n\}$. The effectiveness is proved as follows: Suppose $\sum_{i=n-k+1}^{n} y_i > v(k)$, then we get the contradiction

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n-k} y_i + \sum_{i=n-k+1}^{n} y_i > (n-k) \cdot (1-v(k))/(n-k) + v(k) = 1$$

since $y_{n-k+1} > x_{n-k+1} \geq (1-v(k))/(n-k)$. The rest of the proof is exactly the same as in Case (i). $\square$

5.1 The Uniqueness of Lucas' Symmetric Stable Set for (n; n-1) Games

In Chapter III, we reviewed Lucas' theorem which gives a symmetric stable set for (n; n-1) games. The next theorem demonstrates the uniqueness of this symmetric stable set.

Theorem 5.1: Let K_{sym} be defined as in Theorem 3.2. Then K_{sym} is the unique symmetric stable set.

Proof: It is sufficient to show the uniqueness. Let K be any symmetric stable set and take any $x \in K\text{-}C$. Then the follwoing claim holds.

Claim: For any $i = 1,2,\ldots,[[n/2]]$, if $x_{2i-1} > 1 - v(n-1)$, then $x_{2i-1} = x_{2i}$.

If this claim is true, then $x \in K_{sym} - C$ and thus $K \subseteq K_{sym}$. Therefore the uniqueness of K_{sym} is achieved by internal stability of K_{sym} and external stability of K.

Proof of Claim: This proof will be proceeded by induction. In the case where $i = 1$, this claim follows from the previous lemma. Assume that the claim holds for $i \leq k$. Suppose $x_{2(k+1)-1} > 1 - v(n-1)$ and $x_{2(k+1)-1} > x_{2(k+1)}$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i \neq 2(k+1) - 1 \\ x_{2(k+1)-1} - (n-1)\varepsilon & \text{for } i = 2(k+1) - 1 \end{cases}$$

where $0 < \varepsilon < \min\{(x_{2(k+1)-1} - (1 - v(n-1)))/(n-1),\ (x_{2(k+1)-1} - x_{2(k+1)})/n\}$.

Then y dom x via $N - \{2(k+1) - 1\}$. Hence $y \notin K$ and thus there is some $z \in K$ which dominates y. Now we have $x_{2k-1} > 1 - v(n-1)$ since $x_{2(k+1)-1} > 1 - v(n-1)$. Thus by the induction hypothesis, $x_1 = x_2 \geq \ldots \geq x_{2k-1} = x_{2k}$ or $y_1 = y_2 \geq \ldots \geq y_{2k-1} = y_{2k}$. Also we have $z_1 = z_2 \geq \ldots \geq z_{2k-1} = z_{2k}$ since z dom y and $y_{2(k+1)-1} > 1 - v(n-1)$. Suppose z dom y via $(N - \{i(0)\})_z|(N - \{i\})_y$. If $i(0) \geq 2(k+1) - 1$,

then z dom x via $(N - \{i(0)\})_z | (N - \{2(k+1) - 1\})_x$. If $i(0) \leq 2k$, then $Z_{2(k+1)-1} > y_{2(k+1)-1} > 1 - v(n-1)$. Hence $N - \{2(k+1) - 1\}$ is effective for z and thus z dom x via $N - \{2(k+1) - 1\}$. In either case we obtain z dom x which contradicts the fact that $x,z \in K$. □

5.2 Finite Symmetric Stable Sets

For $(n;k)$ games with $k \leq (n+1)/2$, if $v(k)$ is "large enough", then there exists a unique finite symmetric stable set.

Theorem 5.2: Consider $(n;k)$ games with $k \leq (n+1)/2$. Then the set $K_{sym} = \langle(\underbrace{1-(n-k+1),\ldots,1/(n-k+1)}_{n-k+1},\underbrace{0,\ldots,0}_{k-1})\rangle$ is the unique symmetric stable set if and only if $v(k) \geq k/(n-k+1)$.

Proof: Sufficiency: It is easy to show that K_{sym} is a symmetric stable set. The uniqueness of K_{sym} is proved as follows.

Suppose K is a symmetric stable set and take any $x \in K$. Then by Lemma 5.1, $x_1 = \ldots = x_{n-k+1} > (1 - v(k))/(n-k)$. Now assume $x_{n-k+1} < 1/(n-k+1)$ and define y by

$$y_i = \begin{cases} (1/(n-k+1)) - \varepsilon & \text{for } i = 1,\ldots,n-k+1 \\ \varepsilon/(k-1) & \text{for } i = n-k+2,\ldots,n \end{cases}$$

where $0 < \varepsilon < (1/(n-k+1)) - x_{n-k+1}$. Then y dom x via $\{1,\ldots,k\}$ since $k \leq (n+1)/2$ and $v(k) \geq k/(n-k+1)$. Hence $y \notin K$ and thus there is some $z \in K$ which dominates y. Assume z dom y via $\{i(1),\ldots,i(k)\}_z | \{n-k+1,\ldots,n\}_y$, then $z_{i(1)} > y_{n-k+1} > x_{n-k+1}$.

Again from Lemma 5.1, $z_1 = \ldots = z_{n-k+1}$ and moreover $z_{n-k+1} \leq 1/(n-k+1)$. In fact, if $z_{n-k+1} > 1/(n-k+1)$ then we have the contradiction $\sum_{i=1}^{n} z_i > 1$. Thus z dom x via $\{1,\ldots,k\}$ which contradicts the fact that $x,z \in K$. Hence $x_{n-k+1} \geq 1/(n-k+1)$ which implies that x must be of the form $(\underbrace{1/(n-k+1),\ldots,1/(n-k+1)}_{n-k+1},\underbrace{0,\ldots,0}_{k-1})$, i.e., $x \in K_{sym}$.

Therefore $K \subseteq K_{sym}$ from which the uniqueness of K_{sym} follows.

Necessity: This is clear. In fact, if $v(k) < k/(n-k+1)$, then K_{sym} does not satisfy external stability.

In the following sections, symmetric stable sets for $(n;2)$, $(n;3)$ and $(n;4)$ games will be obtained even when the condition in Theorem 5.1 is not fulfilled.

5.3 (n;2) Games

Theorem 5.3: Assume $v(2) < 2/(n-1)$. Then the set

$$K_{sym} = \langle\{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-1} \geq (1-v(2))/(n-2) \geq x_n;$$

$$x_{n-1} + x_n < v(2);\ \ x_{n-2} + x_{n-1} \geq v(2)\}\rangle \cup C$$

is the unique symmetric stable set for $(n;2)$ games.

Proof: We will first show that K_{sym} is a stable set. When $C \neq \emptyset$, this was already proved in Chapter IV. Thus we assume $C = \emptyset$, i.e., $v(2) > 2/n$.

Internal stability: Take any $x,y \in K_{sym}$ and assume x dom y via $S_x | \{n-1,n\}_y$. Here S_x should be of the form $\{i,n\}_x$ where $i \in \{1,\ldots,n-1\}$. In fact, if S_x consists of two indices, say k and ℓ, from $\{1,\ldots,n-1\}$, then by the effectiveness of S_x we have $x_k = x_\ell = v(2)/2$ and thus x cannot dominate y via S_x. Therefore we get the contradiction $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$.

External stability: We first note that for any $x \in A$ $x_{n-1} + x_n < v(2)$, since $v(2) > 2/n$. Take any $x \in A - K_{sym}$. If $x_{n-1} < v(2)/2$, then $y = (v(2)/2,\ldots,v(2)/2,\ 1-v(2)\cdot(n-1)/2)$ dominates x via $\{n-2, n-1\}_y | \{n-1, n\}_x$. Clearly $y \in K_{sym}$. Thus we assume $x_{n-1} > v(2)/2$. Suppose $x_i > x_{i+1}$ for some $i = 1,\ldots,n-2$, then $\sum_{i=1}^{n-1} x_i > (n-1)x_{n-1}$. Define y by

$$y_i = \begin{cases} x_{n-1} + \varepsilon & \text{for } i = 1,\ldots,n-1 \\ x_i + \varepsilon & \text{for } i = n \end{cases}$$

where $n\varepsilon = \sum_{i=1}^{n-1} x_i - (n-1)x_{n-1}$. Then $y \in K_{sym}$ and y dom x via $\{n-1, n\}$.

Uniqueness: Suppose K is a symmetric stable set and choose any $x \in K-C$.

Case (i) $C \neq \emptyset$: From Lemma 5.1, $x_1 = \ldots = x_{n-1} \geq (1-v(2))/(n-2)$. In the same way as in the proof of Theorem 5.1, we obtain $x_{n-1} \geq v(2)/2$. Thus $x \in K_{sym}$ since $x_{n-1} + x_n < v(2)$. Hence $K \subseteq K_{sym}$.

Case (ii) $C \neq \emptyset$: If $x \in K-V$, then we must have

$x_1 = \ldots = x_{n-1} > (1-v(2))/(n-2)$ and $x_{n-1} + x_n < v(2)$. Hence $x \in K_{sym} - C$ since $x_{n-2} + x_{n-1} > 2 \cdot (1-v(2))/(n-2) \geq v(2)$. Thus $K-C \subseteq K_{sym}-C$ which implies the uniqueness of K_{sym}. □

5.4 (n;3) <u>Games</u>

<u>Theorem 5.4</u>: Assume $n \geq 5$ and $v(3) < 3/(n-2)$. Define

$$K_1 = \langle\{x \in \lceil A\rceil \mid x_1 = \ldots = x_{n-2} > (1-v(3))/(n-3) \geq x_{n-1} \geq x_n;$$
$$x_{n-2} + x_{n-1} + x_n < v(3);\ x_{n-3} + x_{n-2} + x_n \geq v(3)\}\rangle$$

and

$$K_2 = \langle\{x \in \lceil A\rceil \mid x_1 = \ldots = x_{n-2} > (1-v(3))/(n-3) \geq x_{n-1} = x_n;$$
$$x_{n-3} + x_{n-2} + x_n < v(3);\ x_{n-4} + x_{n-3} + x_{n-2} \geq v(3)\}\rangle.$$

Then $K_{sym} = K_1 \cup K_2 \cup C$ is the unique symmetric stable set for (n;3) games.

<u>Proof</u>: <u>Internal stability</u>: Take any $x,y \in K_{sym}$ and assume $x \text{ dom } y$ via $S_x \mid \{n-2, n-1, n\}_y$.

<u>Case (i)</u> $x \in C$, $y \in K_1 \cup K_2$: Without loss of generality, assume $S_x = \{n-2, n-1, n\}_x$. Then $x_{n-2} > y_{n-2} > (1-v(3))/(n-3)$ and thus we get the contradiction $\sum_{i=1}^{n} x_i > 1 - v(3) + v(3) = 1$.

<u>Case (ii)</u> $x,y \in K_1 \cup K_2$: If $S_x = \{n-2, n-1, n\}_x$ then obviously $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$, since $\sum_{i \in S_x} x_i \geq v(4)$ and thus $\sum_{i \notin S_x} x_i \leq 1 - v(4)$ for all $x \in K_1 \cup K_2$. Thus it is sufficient to consider the following two cases: (a). $S_x = \{n-3, n-2, n\}_x$ and (b). $S_x = \{n-3, n-2, n-1\}_x$.

<u>(ii-I)</u> $x,y \in K_1$: (a). Since $S_x = \{n-3, n-2, n\}_x$, $x_{n-3} = x_{n-2} > y_{n-2}$, $x_n > y_n$ and $x_{n-3} + x_{n-2} + x_n = v(3)$. Hence we get the contradiction

$$\sum_{i=1}^{n} x_i = \sum_{i \neq n-3,n-2,n} x_i + x_{n-3} + x_{n-2} + x_n$$

$$> 1 - v(3) + y_{n-3} + y_{n-2} + y_n \geq \sum_{i=1}^{n} y_i.$$

The second inequality follows from the definition of K_1.

(b). We can easily obtain a similar contradiction in this case.

<u>(ii-II)</u> $x \in K_1$, $y \in K_2$: (a). Since $S_x = \{n-3, n-2, n\}_x$, $x_{n-3} = x_{n-2} > y_{n-2}$ and $(x_{n-1} \geq)\ x_n > y_n\ (= y_{n-1})$. Hence $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$.

(b). Since $v(3) \leq x_{n-3} + x_{n-2} + x_n \leq x_{n-3} + x_{n-2} + x_{n-1} \leq v(3)$, $x_{n-1} = x_n$. Thus $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$.

<u>(ii-III)</u> $x \in K_2$, $y \in K_1$: This is the same as (a) of (ii-I).

<u>(ii-IV)</u> $x,y \in K_2$: Since $x_{n-1} = x_n$ and $y_{n-1} = y_n$, we get $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$.

<u>External stability</u>: Take any $x \in A - K_{sym}$.

<u>Case (i)</u> $C = \emptyset$: We first note that $x_{n-2} \geq v(3)/3$. In fact, if not, then $y = (v(3)/3, \ldots, v(3)/3, \ (1-v(3)\cdot(n-2)/3)/2, \ (1-v(3)\cdot(n-2)/3)/2)$ dominates x via $\{n-4, n-3, n-2\}_y | \{n-2, n-1, n\}_x$. Clearly $y \in K_1$. Let $n\varepsilon^1 = \sum_{i=1}^{n-2} x_i - (n-2)x_{n-2}$.

<u>(i-I)</u> $\varepsilon^1 > 0$: Define y by

$$y_i = \begin{cases} x_{n-2} + \varepsilon^1 & \text{for } i = 1,\ldots,n-2 \\ x_i + \varepsilon^1 & \text{for } i = n-1,n. \end{cases}$$

Then $y_{n-2} + y_{n-1} + y_n < v(3)$. If $y_{n-3} + y_{n-2} + y_n \geq v(3)$, then $y \in K_1$ and y dom x via $\{n-2, n-1, n\}$. Thus we assume $y_{n-3} + y_{n-2} + y_n < v(3)$. If $2y_{n-3} + 2y_{n-2} + y_{n-1} + y_n \geq 2v(3)$, then define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\ldots,n-2 \\ y_i - \varepsilon & \text{for } i = n-1 \\ y_i + \varepsilon & \text{for } i = n \end{cases}$$

where $\varepsilon > 0$, $y'_{n-3} + y'_{n-2} + y'_{n-1} \geq v(3)$ and $y'_{n-3} + y'_{n-2} + y'_n = v(3)$. Then $y' \in K_1$ and y' dom x via $\{n-3, n-2, n\}_{y'} | \{n-2, n-1, n\}_x$. If $2y_{n-3} + 2y_{n-2} + y_{n-1} + y_n < 2v(3)$, then define y'' by

$$y''_i = \begin{cases} y_i & \text{for } i = 1,\ldots,n-2 \\ (y_{n-1} + y_n)/2 & \text{for } i = n-1,n. \end{cases}$$

Then $y'' \in K_2$. $y''_{n-1} = y''_n \leq (1-v(3))/(n-3)$ is proved as follows.

Suppose otherwise, then we obtain the contradiction

$$\sum_{i=1}^{n} y_i'' = \sum_{i=1}^{n-2} y_i'' + y_{n-1}'' + y_n'' > (n-2)\cdot v(3)/3 + 2\cdot(1-v(3))/(n-3)$$

$$= (n(n-5)\cdot v(3)+6)/3(n-3) \geq 1$$

since $v(3) > 3/n$ and $n \geq 5$.

Hence $y'' \in K_2$ and y'' dom x via $\{n-3, n-2, n\}_{y''}|\{n-2, n-1, n\}_x$.

<u>(i-II)</u> $\varepsilon^1 = 0$: Since $x \notin K_{sym}$, we must have $x_{n-3} + x_{n-2} + x_n < v(3)$ and $x_{n-1} > x_n$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon' & \text{for } i = 1,\ldots,n-2 \\ x_n + \varepsilon'' & \text{for } i = n-1,n \end{cases}$$

where $(n-2)\varepsilon' + 2\varepsilon'' = x_{n-1} - x_n$ and ε' is sufficiently small so that $y_{n-3} + y_{n-2} + x_n < v(3)$.

If $y_{n-3} + y_{n-2} + y_n < v(3)$ then $y \in K_2$ and y dom x via $\{n-3, n-2, n\}_y|\{n-2, n-1, n\}_x$. Thus we assume $y_{n-3} + y_{n-2} + y_n > v(3)$. Define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,n-2 \\ y_i + \varepsilon & \text{for } i = n-1 \\ y_i - \varepsilon & \text{for } i = n \end{cases}$$

where $\varepsilon > 0$ and $y'_{n-3} + y'_{n-2} + y'_n = v(3)$. Then $y' \in K_1$ and y' dom x via $\{(n-3, n-2, n\}_{y'} | \{n-2, n-1, n\}_x$ since $y'_{n-3} + y'_{n-2} + y'_n = v(3) > y_{n-3} + y_{n-2} + x_n$ implies $y'_n > x_n$.

Case (ii) $C \neq \emptyset$: If $x_{n-3} < (1-v(3))/(n-3)$, then there is some $y \in C$ such that y dom x. Thus $x_{n-3} \geq (1-v(3))/(n-3)$. Moreover $x_1 \geq (1-v(3))/(n-3)$ since $x \notin C$. From these two facts, $x \in \text{Dom } K_{sym}$ is verified in a manner similar to Case (i).

Thus we have completed the proof of internal and external stability of K_{sym}.

Uniqueness: Suppose K is a symmetric stable set and take any $x \in K$. Then from Lemma 5.1, $x_1 = \ldots = x_{n-2} \geq (1-v(3))/(n-3)$.

Case (i) $C = \emptyset$: We must have $x_{n-2} \geq v(3)/3$. If $x_{n-3}+x_{n-2}+x_n \geq v(3)$, then $x \in K_1$. Thus we assume $x_{n-3} + x_{n-2} + x_n < v(3)$. Now we will show $x_{n-1} = x_n$ and thus $x \in K_2$. Suppose $x_{n-1} > x_n$ and define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i = 1,\ldots,n-2,n \\ x_{n-1} - (n-1)\varepsilon & \text{for } i = n-1 \end{cases}$$

where

$$0 < \varepsilon < \min((x_{n-1} - x_n)/n, \ (v(3) - (x_{n-3} + x_{n-2} + x_n))/3).$$

Then y dom x via $\{n-3, n-2, n\}_y | \{n-2, n-1, n\}_x$. Hence $y \notin K$ and thus there is some $z \in K$ such that z dom y. Suppose z dom y via

$\{i(1),i(2),i(3)\}_z|\{n-2,\ n-1,\ n\}_x$. Then at least two of $i(1)$, $i(2)$ and $i(3)$ must belong to $\{1,\ldots,n-2\}$ since $z_1 = \ldots = z_{n-2}$ and $y_1 = \ldots = y_{n-2}$. Without loss of generality, assume $i(1) = n-3$ and $i(2) = n-2$ then $z_{i(1)} = z_{i(2)} > y_{n-2} > x_{n-2} = x_{n-3}$ and $z_{i(3)} > y_n > x_n$. Thus $z \text{ dom } x$ via $\{i(1),i(2),i(3)\}_z|\{n-3,\ n-2,\ n\}_x$ which contradicts the fact that $x,z \in K$.

Therefore we obtain $x \in K_{sym}$ and thus $K \subseteq K_{sym}$.

<u>Case (ii)</u> $C \neq \emptyset$: We must have $x_1 > (1-v(3))/(n-3)$ and $x_{n-2} \geq (1-v(3))/(n-3)$. Thus in the same way as above we get $K-C \subseteq K_{sym}-C$. Hence $K \subseteq K_{sym}$.

In either case, we obtain $K \subseteq K_{sym}$ which implies the uniqueness of K_{sym}. □

In the case where $n = 4$, Theorems 3.2 and 5.2 show that the unique symmetric stable set is given by

$$K_{sym} = K_1 \cup K_2 \cup C$$

where

$$K_1 = \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq 1-v(3) \geq x_3 \geq x_4\}\rangle$$

and

$$K_2 = \langle\{x \in \lceil A \rceil | x_1 = x_2 \geq x_3 = x_4 > 1-v(3)\}\rangle.$$

5.5 (n;4) <u>Games</u>

<u>Theorem 5.5</u>: Assume $n \geq 7$ and $v(4) < 4/(n-3)$. Define

$$K_1 = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-3} \geq (1-v(4))/(n-4) \geq x_{n-2} \geq x_{n-1} > x_n;$$

$$x_{n-3} + x_{n-2} + x_{n-1} + x_n \leq v(4);\ x_{n-4} + x_{n-3} + x_{n-1} + x_n \geq v(4)\} \rangle,$$

$$K_2 = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-3} \geq (1-v(4))/(n-4) \geq x_{n-2} = x_{n-1} \geq x_n;$$

$$x_{n-4} + x_{n-3} + x_{n-1} + x_n < v(4);\ x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1} \geq v(4);$$

$$x_{n-5} + x_{n-4} + x_{n-3} + x_n \geq v(4)\} \rangle,$$

$$K_3 = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-3} \geq (1-v(4))/(n-4) \geq x_{n-2} = x_{n-1} \geq x_n;$$

$$x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1} \leq v(4);\ x_{n-5} + x_{n-4} + x_{n-3} + x_n = v(4)\} \rangle,$$

$$K_4 = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-3} \geq (1-v(4))/(n-4) \geq x_{n-2} = x_{n-1} \geq x_n = 0;$$

$$x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1} < v(4);\ x_{n-5} + x_{n-4} + x_{n-3} \geq v(4)\} \rangle$$

and

$$K_5 = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-3} \geq (1-v(4))/(n-4) \geq x_{n-2} = x_{n-1} = x_n;$$

$$x_{n-5} + x_{n-4} + x_{n-3} + x_n < v(4);\ x_{n-6} + x_{n-5} + x_{n-4} + x_{n-3} \geq v(4)\} \rangle,$$

Then $K_{sym} = (\bigcup_{i=1}^{5} K_i) \cup C$ is a symmetric stable set.

<u>Proof</u>: <u>Internal stability</u>: Take any $x, y \in K_{sym}$ and assume x dom y via $S_x | \{n-3, n-2, n-1, n\}_y$.

<u>Case (i)</u> $x \in C$, $y \in \bigcup_{i=1}^{5} K_i$: Without loss of generality, assume $S_x = \{n-3, n-2, n-1, n\}_x$, then $x_{n-3} \geq y_{n-3} \geq (1-v(4))/(n-4)$. Thus we

have the contradiction $\sum_{i=1}^{n} x_i > 1-v(4) + v(4) = 1$.

<u>Case (ii)</u> $x,y \in \bigcup_{i=1}^{5} K_i$: If $S_x = \{n-3, n-2, n-1, n\}_x$, then obviously $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. If $S_x \subseteq \{1,\ldots,n-3\}_x$ then we again get $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$ since $\sum_{i\in S_x} x_i \geq v(4)$ and thus $\sum_{i\notin S_x} x_i \leq 1-v(4)$. Thus it suffices to consider the following six cases: (a). $S_x = \{n-4,n-3,n-1,n\}_x$. (b). $S_x = \{n-4, n-3, n-2, n\}_x$, (c). $S_x = \{n-4, n-3, n-2, n-1\}_x$, (d). $S_x = \{n-5, n-4, n-3, n\}_x$, (e). $S_x = \{n-5, n-4, n-3, n-1\}_x$ and (f). $S_x = \{n-5, n-4, n-3, n-2\}_x$.

(ii-I) $x,y \in K_1$: (a). Here $x_{n-4} = x_{n-3} > y_{n-3}$, $x_{n-1} > y_{n-1}$, $x_n > y_n$ and $x_{n-4} + x_{n-3} + x_{n-1} + x_n = v(4)$. Hence we get the contradiction

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-5} x_i + x_{n-2} + x_{n-4} + x_{n-3} + x_{n-1} + x_n$$

$$> 1 - v(4) + y_{n-4} + y_{n-3} + y_{n-1} + y_n \geq \sum_{i=1}^{n} y_i .$$

The second inequality follows from the definition of K_1. (b) to (f). In a similar manner, we can obtain contradictions for these cases.

<u>(ii-II)</u> $x \in K_1$, $y \in K_2$: (a). $x_{n-4} = x_{n-3} > y_{n-3}$, $(x_{n-2} \geq)\ x_{n-1} > y_{n-1}\ (= y_{n-2})$ and $x_n > y_n$. Hence $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (b). Since $v(4) \leq x_{n-4} + x_{n-3} + x_{n-1} + x_n \leq x_{n-4} + x_{n-3} + x_{n-2} + x_n \leq v(4)$, we must have $x_{n-1} = x_n$. Thus $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (c) to (f). These cases are the same as (b).

<u>(ii-III)</u> $x \in K_1$, $y \in K_i$ (i=3,4,5): The same as (ii-II).

<u>(ii-IV)</u> $x \in K_2$, $y \in K_1$: The same as (ii-I).

<u>(ii-V)</u> $x,y \in K_2$: (a). $x_{n-4} = x_{n-3} > y_{n-3}$, $(x_{n-2} =)\ x_{n-1} > y_{n-1}\ (= y_{n-2})$ and $x_n > y_n$. Hence $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (b). The same as (a).
(c). $x_{n-4} = x_{n-3} > y_{n-3}$, $x_{n-2} = x_{n-1} > y_{n-1}\ (= y_{n-2})$ and $x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1} = v(4)$. Hence

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-5} x_i + x_n + x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1}$$

$$> 1 - v(4) + y_{n-4} + y_{n-3} + y_{n-2} + y_{n-1} \geq \sum_{i=1}^{n} y_i .$$

(d) to (f). The same as (c).

<u>(ii-VI)</u> $x \in K_2$, $y \in K_3$: (a), (b). The same as (a) in (ij-V).
(c). $x_{n-4} = x_{n-3} > y_{n-3}$, $x_{n-2} = x_{n-1} > y_{n-1}\ (= y_{n-2})$ and $x_{n-5} + x_{n-4} + x_{n-3} + x_n \geq v(4) = y_{n-5} + y_{n-4} + y_{n-3} + y_n$. Thus $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (d). $x_{n-5} = x_{n-4} = x_{n-3} > y_{n-3}$ and $x_n > y_n$. Thus $x_{n-5} + x_{n-4} + x_{n-3} + x_n > y_{n-5} + y_{n-4} + y_{n-3} + y_n = v(4)$ which contradicts the effectiveness of S_x. (e), (f). The same as (d).

<u>(ii-VII)</u> $x \in K$, $y \in K_4$: (a), (b). The same as (a) in (ii-IV).
(c). $x_{n-4} = x_{n-3} > y_{n-3}$, $x_{n-2} = x_{n-1} > y_{n-1}\ (= y_{n-2})$ and $x_n \geq y_n = 0$. Hence $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (d) to (f). The same as (d) in (ii-VI).

<u>(ii-VIII)</u> $x \in K_2$, $y \in K_5$: (a), (b). The same as (a) in (ii-V).
(c). The same as (c) in (ii-VI). (d). $x_{n-5} = x_{n-4} = x_{n-3} > y_{n-3}$ and $(x_{n-2} \geq x_{n-1} \geq)\ x_n \geq y_n\ (= y_{n-1} = y_{n-2})$. Hence $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$.

(e). Since $v(4) \leq x_{n-5} + x_{n-4} + x_{n-3} + x_n \leq x_{n-5} + x_{n-4} + x_{n-3} + x_{n-1} \leq v(4)$, we must have $x_{n-1} = x_n$. Thus $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (f). The same as (e).

<u>(ii-IX)</u> $x \in K_3$, $y \in K_1$: The same as (ii-I).

<u>(ii-X)</u> $x \in K_3$, $y \in K_2$: The same as (ii-V).

<u>(ii-XI)</u> $x,y \in K_3$: (a), (b). The same as (a) in (ii-IV). (c). $x_{n-4} = x_{n-3} > y_{n-4} = y_{n-3}$, $x_{n-2} = x_{n-1} > y_{n-2} = y_{n-1}$ and $x_{n-5} + x_{n-4} + x_{n-3} + x_n = v(4) = y_{n-5} + y_{n-4} + y_{n-3} + y_n$. Thus $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (d) to (f). The same as (d) in (ii-IV).

<u>(ii-XII)</u> $x \in K_3$, $y \in K_4$: (a), (b). The same as (a) in (ii-V). (c). The same as (d) in (ii-VII). (d) to (f). The same as (d) in (ii-VI).

<u>(ii-XIII)</u> $x \in K_3$, $y \in K_5$: (a), (b). The same as (a) in (ii-V). (c). The same as (c) in (ii-VI). (d). The same as (d) in (ii-VIII). (e), (f). The same as (e) in (ii-VIII).

<u>(ii-XIV)</u> $x \in K_4$, $y \in K_1$: The same as (ii-I).

<u>(ii-XV)</u> $x \in K_4$, $y \in K_2$: The same as (ii-V).

<u>(ii-XVI)</u> $x \in K_4$, $y \in K_3$: The same as (ii-XI).

<u>(ii-XVII)</u> $x,y \in K_4$: Since $x_n = 0$, we can ignore (a), (b) and (d). (c). Here $x_{n-4} = x_{n-3} > y_{n-3}$ and $x_{n-2} = x_{n-1} > y_{n-1}$ $(= y_{n-2})$. Hence $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$. (e), (f). $v(4) \leq x_{n-5} + x_{n-4} + x_{n-3} \leq x_{n-5} + x_{n-4} + x_{n-3} + x_i \leq v(4)$ for $i = n-1, n-2$. Thus we must have

$x_{n-1} = x_{n-2} = 0.$

<u>(ii-XVIII)</u> $x \in K_4$, $y \in K_5$: The same as (ii-XVII).

<u>(ii-XIX)</u> $x \in K_5$, $y \in K_i$ $(i=1,2,3,4)$: The same as (ii-XVII).

<u>(ii-XX)</u> $x,y \in K_5$: Obviously $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$.

<u>External stability</u>: Take any $x \in A\text{-}K_{sym}$.

<u>Case (i)</u>: $C = \emptyset$: If $x_{n-3} < v(4)/4$ then $y = (v(4)/4,\ldots,v(4)/4,$ $(1-v(4)\cdot(n-3)/4)/3,$ $(1-v(4)\cdot(n-3)/4)/3,$ $(1-v(4)\cdot(n-3)/4)/3)$ $(\in K_5)$ dominates x via $\{n-6, n-5, n-4, n-3\}_y|\{n-3, n-2, n-1, n\}_x$. Thus we assume $x_{n-3} \geq v(4)/4$. Let $n\varepsilon^1 = \sum_{i=1}^{n-3} x_i - (n-3)x_{n-3}$.

<u>(i-I)</u> $\varepsilon^1 > 0$: Define y by

$$y_i = \begin{cases} x_{n-3} + \varepsilon^1 & \text{for } i = 1,\ldots,n-3 \\ x_i + \varepsilon^1 & \text{for } i = n-2,\ n-1,\ n. \end{cases}$$

Then $y_{n-3} + y_{n-2} + y_{n-1} + y_n < v(4)$. If $y_{n-4} + y_{n-3} + y_{n-1} + y_n \geq v(4)$, then $y \in K_1$ and y dom x via $\{n-3, n-2, n-1, n\}$. Thus we assume $y_{n-4} + y_{n-3} + y_{n-1} + y_n > v(4)$. If $2y_{n-4} + 2y_{n-3} + y_{n-2} + y_{n-1} + 2y_n \geq 2v(4)$, then define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,n-3 \\ y_i - \varepsilon & \text{for } i = n-2 \\ y_i + \varepsilon & \text{for } i = n-1 \\ y_i & \text{for } i = n. \end{cases}$$

where $y'_{n-4} + y'_{n-3} + y'_{n-2} + y'_n \geq v(4)$ and $y'_{n-4} + y'_{n-3} + y'_{n-1} + y'_n = v(4)$.
Then $y' \in K_1$ and y dom x via $\{n-4, n-3, n-1, n\}_y | \{n-3, n-2, n-1, n\}_x$.
If $2y_{n-4} + 2y_{n-3} + y_{n-2} + y_{n-1} + 2y_n < 2v(4)$, then define z by

$$z_i = \begin{cases} y_i & \text{for } i = 1,\dots,n-3 \\ (y_{n-2} + y_{n-1})/2 & \text{for } i = n-2, n-1 \\ y_i & \text{for } i = n \end{cases}$$

Now the following two cases must be considered.

<u>(a)</u> $z_{n-5} + z_{n-4} + z_{n-3} + z_n \geq v(4)$: If $z_{n-4} + z_{n-3} + z_{n-2} + z_{n-1} \geq v(4)$, then $z \in K_2$ and z dom x via $\{n-4, n-3, n-1, n\}_z | \{n-3, n-2, n-1, n\}_x$.
Thus we assume $z_{n-4} + z_{n-3} + z_{n-2} + z_{n-1} < v(4)$:

<u>(a-I)</u> $z_{n-5} + z_{n-4} + z_{n-3} < v(4)$: If $z_{n-5} + 2z_{n-4} + 2z_{n-3} + z_{n-2} + z_{n-1} + z_n \geq 2v(4)$, then define z' by

$$z'_i = \begin{cases} z_i & \text{for } i = 1,\dots,n-3 \\ z_i + \varepsilon & \text{for } i = n-2, n-1 \\ z_i - 2\varepsilon & \text{for } i = n \end{cases}$$

where $\varepsilon > 0$, $z'_{n-5} + z'_{n-4} + z'_{n-3} + z'_n \geq v(4)$ and $z'_{n-4} + z'_{n-3} + z'_{n-2} + z'_{n-1} = v(4)$.
Then $z'' \in K_2$ and z' dom x via $\{n-4, n-3, n-2, n-1\}_{z'} | \{n-3, n-2, n-1, n\}_x$.
If $z_{n-5} + 2z_{n-4} + 2z_{n-3} + z_{n-2} + z_{n-1} + z_n > 2v(4)$, then define z'' by

$$z_i'' = \begin{cases} z_i & \text{for } i = 1,\dots,n-3 \\ z_i + \varepsilon & \text{for } i = n-2,\ n-1 \\ z_i - 2\varepsilon & \text{for } i = n \end{cases}$$

where $\varepsilon > 0$, $z''_{n-5} + z''_{n-4} + z''_{n-3} + z''_{n} = v(4)$ and $z''_{n-4} + z''_{n-3} + z''_{n-2} + z''_{n-1} < v(4)$.

Then $z'' \in K_3$ and z'' dom x via $\{n-4,n-3,n-2,n-1\}_{z''} | \{n-3,n-2,n-1,n\}_x$.

<u>(a-II)</u> $z_{n-5} + z_{n-4} + z_{n-3} \geq v(4)$: If $z_n \geq v(4)-(z_{n-4}+z_{n-3}+z_{n-2}+z_{n-1})$, then define z' by

$$z_i' = \begin{cases} z_i & \text{for } i = 1,\dots,n-3 \\ z_i + \varepsilon & \text{for } i = n-2,\ n-1 \\ z_i - 2\varepsilon & \text{for } i = n \end{cases}$$

where $\varepsilon > 0$, $z'_{n-5} + z'_{n-4} + z'_{n-3} + z'_{n} \geq v(4)$ and $z'_{n-4} + z'_{n-3} + z'_{n-2} + z'_{n-1} = v(4)$.

Then $z' \in K_2$ and z' dom x via $\{n-4, n-3, n-2, n-1\}_{z'} | \{n-3,n-2,n-1,n\}_x$.

If $z_n < v(4) - (z_{n-4} + z_{n-3} + z_{n-2} + z_{n-1})$ then define z'' by

$$z_i'' = \begin{cases} z_i & \text{for } i = 1,\dots,n-3 \\ z_i + z_n/2 & \text{for } i = n-2,\ n-1 \\ 0 & \text{for } i = n \end{cases}$$

Then $z'' \in K_4$ and z'' dom x via $\{n-4,n-3,n-2,n-1\}_{z''} | \{n-3,n-2,n-1,n\}_x$.

<u>(b)</u> $z_{n-5} + z_{n-4} + z_{n-3} + z_n < v(4)$: Let $3\varepsilon^2 = \sum_{i=n-2}^{n} z_i - 3z_n$. If $\varepsilon^2 = 0$ then $z \in K_5$ and $z \operatorname{dom} x$ via $\{n-5, n-4, n-4, n\}_z | \{n-3, n-2, n-1, n\}_x$. Here $z \in K_5$ is proved as follows: It suffices to show that $z_{n-2} \leq (1-v(4))/(n-4)$. Assume otherwise, then we have the contradiction $\sum_{i=1}^{n} z_i > (n-3) \cdot v(4)/4 + 3 \cdot (1-v(4))/(n-4) \geq 1$ since $v(4) \geq n$ and $n \geq 7$. Hence $z_{n-2} \leq (1-v(4))/(n-4)$. Thus we assume $\varepsilon^2 > 0$ and define z' by

$$z'_i = \begin{cases} z_i & \text{for } i = 1,\ldots,n-3 \\ z_i + \varepsilon^2 & \text{for } i = n-2, n-1, n \end{cases}$$

If $z'_{n-5} + z'_{n-4} + z'_{n-3} + z'_n < v(4)$ then $z' \in K_5$ and $z' \operatorname{dom} x$ via $\{n-5, n-4, n-3, n\}_{z'} | \{n-3, n-2, n-1, n\}_x$. Thus we assume $z'_{n-5} + z'_{n-4} + z'_{n-3} + z'_n > v(4)$ and define z'' by

$$z''_i = \begin{cases} z'_i (= z_i) & \text{for } i = 1,\ldots,n-3 \\ z'_i + \varepsilon & \text{for } i = n-2, n-1 \\ z'_i - 2\varepsilon & \end{cases}$$

where $\varepsilon > 0$ and $z''_{n-5} + z''_{n-4} + z''_{n-3} + z''_n = v(4)$. Then $z'' \in K_2$ or K_3 and $z'' \operatorname{dom} x$ via $\{n-5, n-4, n-3, n\}_{z''} | \{n-3, n-2, n-1, n\}_x$, since $z''_n > z'_n$.

<u>(i-II)</u> $\varepsilon^1 = 0$: Since $x \notin K_{sym}$, $x_{n-4} + x_{n-3} + x_{n-1} + x_n < v(4)$. If $x_{n-2} > x_{n-1}$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i = 1,\dots,n-3 \\ x_i - (n-1)\varepsilon & \text{for } i = n-2 \\ x_i + \varepsilon & \text{for } i = n-1,\ n \end{cases}$$

where ε is sufficiently small so that $y_{n-4} + y_{n-3} + y_{n-1} + y_n < v(4)$ and $y_{n-2} > y_{n-1}$. Then in the same manner as in (i-I), we obtain $x \in \text{Dom } K_{sym}$. Thus we assume $x_{n-2} = x_{n-1}$ and consider the following two cases.

<u>(a)</u> $x_{n-5} + x_{n-4} + x_{n-3} + x_n \geq v(4)$: Since $x \notin K_{sym}$, we obtain $x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1} < v(4)$, $x_{n-5} + x_{n-4} + x_{n-3} + x_n > v(4)$ and $x_n > 0$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i = 1,\dots,n-1 \\ x_i - (n-1) & \text{for } i = n \end{cases}$$

where ε is sufficiently small so that $x_{n-4} + x_{n-3} + x_{n-2} + x_{n-1} < v(4)$, $x_{n-5} + x_{n-4} + x_{n-3} + x_n > v(4)$ and $x_n > 0$. Then in the same manner as in (i-I), we obtain $x \in \text{Dom } K_{sym}$.

<u>(b)</u> $x_{n-5} + x_{n-4} + x_{n-3} + x_n < v(4)$: Since $x \notin K_{sym}$, we must have $x_{n-1} > x_n$. Let $\varepsilon^3 = \sum_{i=n-2}^{n} x_i - 3x_n$ and define y by

$$y_i = \begin{cases} x_i + \varepsilon' & \text{for } i = 1,\dots,n-3 \\ x_i + \varepsilon'' & \text{for } i = n-2,\ n-1,\ n \end{cases}$$

where $\varepsilon', \varepsilon'' > 0$, $(n-3)\varepsilon' + 3\varepsilon'' = \varepsilon^3$ and ε' is sufficiently

small so that $y_{n-5} + y_{n-4} + y_{n-3} + x_n < v(4)$. Then if $y_{n-5} + y_{n-4} + y_{n-3} + y_n < v(4)$, $y \in K_5$ and y dom x via $\{n-5, n-4, n-3, n\}_y | \{n-3, n-2, n-1, n\}_x$. Thus we assume $y_{n-5} + y_{n-4} + y_{n-3} + y_n \geq v(4)$. Define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\ldots,n-3 \\ y_i + \varepsilon & \text{for } i = n-2, n-1 \\ y_i - 2\varepsilon & \text{for } i = n \end{cases}$$

where $y_{n-5} + y_{n-4} + y_{n-3} + y_n = v(4)$. Then $y' \in K_2$ or K_3 and y' dom x via $\{n-5, n-4, n-3, n\}_{y'} | \{n-3, n-2, n-1, n\}_x$ since $y_n > x_n$.

Thus we have completed the proof for Case (i).

Case (ii) $C \neq \emptyset$: If $x_{n-3} < (1-v(4))/(n-4)$, then there is some $y \in C$ which dominates x. Thus we assume $x_{n-3} \geq (1-v(4))/(n-4)$. Moreover we must have $x_1 > (1-v(4))/(n-4)$ since $x \notin C$. Using these two facts, $x \in \text{Dom } K_{sym}$ is proved in a manner quite similar to that in Case (i). □

In the case where $n = 5$, the unique symmetric stable set is given by

$$K_{sym} = K_1 \cup K_2 \cup C$$

where

$$K_1 = \langle \{x \in \lceil A \rceil | x_1 = x_2 \geq 1-v(4) \geq x_3 \geq x_4 \geq x_5 \} \rangle$$

and

$$K_2 = \langle\{x \in [A] | x_1 = x_2 \geq x_3 = x_4 \geq 1-v(4) \geq x_5\}\rangle.$$

We conclude this section by stating the following theorem which gives us a symmetric stable set for (6;4) games.

Theorem 5.6: Define

$$K_1 = \langle\{x \in \lceil A\rceil | x_1 = x_2 = x_3 \geq (1-v(4))/2 \geq x_4 \geq x_5 \geq x_6;$$
$$x_3 + x_4 \geq 1-v(4)\}\rangle,$$

$$K_2 = \langle\{x \in \lceil A\rceil | x_1 = x_2 = x_3 \geq (1-v(4))/2 \geq x_4 = x_5 \geq x_6;$$
$$x_3 + x_4 > 1-v(4); \quad x_3 + x_6 \geq 1-v(4)\}\rangle,$$

$$K_3 = \langle\{x \in \lceil A\rceil | x_1 = x_2 = x_3 \geq (1-v(4))/2 = x_4 = x_5 \geq x_6;$$
$$x_3 + x_6 \geq 1-v(4)\}\rangle,$$

$$K_4 = \langle\{x \in \lceil A\rceil | x_1 = x_2 = x_3 \geq (1-v(4))/2 \geq x_4 = x_5 \geq x_6 = 0;$$
$$x_3 > 1-v(4)\}\rangle$$

and

$$K_5 = \langle\{x \in \lceil A\rceil | x_1 = x_2 = x_3 \geq x_4 = x_5 = x_6 \geq (1-v(4))/2\}\rangle$$

Then $K_{sym} = (\bigcup_{i=1}^{5} K_i) \cup C$ is a symmetric stable set for (6;4) games.

<u>Proof</u>: We omit this proof since it proceeds in the same way as in Theorem 5.5. □

CHAPTER VI

HART'S PRODUCTION GAMES

S. Hart [12] defined a family of symmetric games which reflect some production economics and determined symmetric stable sets for these games under some special conditions. This chapter will be devoted to further study of this class of games. Let us first review his results briefly.

6.1 Preliminaries

Recall Hart games $(n;k)_h$ are given by

$$v(s) = \begin{cases} 0 & \text{for } s < k \\ 1/q & \text{for } k \leq s < 2k \\ \cdots & \cdots \\ j/q & \text{for } jk \leq s < (j+1)k \\ \cdots & \cdots \\ 1 & \text{for } qk \leq s \end{cases}$$

where

$$n = qk + r \quad (q \geq 2 \text{ and } 0 \leq r \leq k-1).$$

Hart gave us the following two theorems and two open questions concerning the games $(n;k)_h$.

Theorem 6.1 (Hart): Define

$$K_h = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-k+1} \geq 1/q \geq x_{n-k+2} = \ldots = x_n\} \rangle .$$

Then K_h is a symmetric stable set for $(n;k)_h$ if and only if $n \geq (q+1)(k-1)$.

Theorem 6.2 (Hart): If $n \geq (q+1)k-3$, then this K_h is the unique symmetric stable set for $(n;k)_h$.

Open question 1: What are the symmetric stable sets, if any, for $(n;k)_h$ when the condition in Theorem 6.1 is not satisfied.

Open question 2: If K_h is the unique symmetric stable set when the condition in Theorem 6.1 is satisfied instead of the condition in Theorem 6.2.

In the following two sections, these two open questions will be investigated.

6.2 Symmetric Stable Sets

For simplicity, let us first assume $r = 0$. Then Theorem 6.1 says that K_h is a stable set if and only if $q \geq k-1$. The next theorem will partly answer the open question 1.

Theorem 6.3: Define

$$K_1 = \langle\{x \in \lceil A\rceil \mid x_1 = \ldots = x_{n-k+1} \geq 1/q(k-1) \geq x_{n-k+2} = \ldots = x_{n-1} \geq x_n = 0\}\rangle$$

and

$$K_2 = \langle\{x \in \lceil A\rceil \mid 1/q(k-1) \geq x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_{n-1} \geq x_n;$$

x is on a curve connecting

$$(\underbrace{1/q(k-1),\ldots,1/q(k-1)}_{n-k+1},\ \underbrace{(k-1-q)/q(k-1)(k-2),\ldots,(k-1-q)/q(k-1)(k-2)}_{k-2},0)$$

with $(1/n,\ldots,1/n)$ where all coordinates $x_1,\ldots,x_n$ vary monotonically; $(k-2)x_1 + 2x_{n-k+2} \leq 1/q$; $(k-1)x_1 + x_n \leq 1/q\}\rangle$.

Then $K_1 \cup K_2$ is a symmetric stable set for $(n;k)_h$ games with $r = 0$ if $[[(k+1)/2]] \leq q \leq k-1$ where $[[k+1/2]]$ is the greatest integer in $(k+1)/2$.

Before proving this theorem, we will state some remarks.

<u>Remarks</u>: (a). If $[[(k+1)/2]] \leq q \leq k-1$, then K_2 is not empty. Namely there always exists at least one line satisfying the condition for K_2. For example, let

$$K_2' = \langle\{x \in \lceil A\rceil \mid 1/q(k-1) \geq x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_{n-1} \geq x_n;$$
$$(k-1)x_1 + x_n = 1/q\}\rangle.$$

Then it is easily shown that K_2' satisfies all the conditions for K_2. In general, there are infinitely many curves satisfying the conditions for K_2 except when $q = [[(k+1)/2]]$ and k is even. (In this case there is only one such line.)

(b). When $q = k-1$, $K_1 \subseteq K_2$. Hence the symmetric stable set consists only of K_2. Hart's stable set K_h for this case is easily shown to be the extreme one of many curves satisfying the conditions for K_2, i.e. the line $(k-1)x_1 + x_n = 1/q$. This fact answers Hart's open question 2 negatively. (The complete answer for this question will be given in the next section.)

Now let us begin to prove Theorem 6.3.

<u>Proof</u>: <u>Internal stability</u>: Pick any two elements, say x and y, in $K_1 \cup K_2$ and assume x dom y via $S_x | \{n-k+1,\ldots,n\}_y$. We will prove only the case $x,y \in K_2$. For other cases, it is easily shown that x cannot dominate y. From the definition of K_2, S_x must be of the form $\{1,\ldots,k-1, n-k+2\}_x$. If this S_x is effective, then $(k-1)x_1 + x_{n-k+2} \leq 1/q$. Hence

$$\begin{aligned}
\sum_{i=1}^{n} x_i &= (n-k+1)x_1 + (k-2)x_{n-k+2} + x_n \\
&= \{(q-1)(k-1) + q\}x_1 + (k-2)x_{n-k+2} + x_n \\
&= (q-1)\{(k-1)x_1 + x_{n-k+2}\} + qx_1 + (k-q-1)x_{n-k+2} + x_n \\
&\leq (q-1)\{(k-1)x_1 + x_{n-k+2}\} + qx_1 + (k-q)x_{n-k+2} \\
&= (q-1)\{(k-1)x_1 + x_{n-k+2}\} + (k-1)x_1 + (q-k+1)x_1 + (k-q)x_{n-k+2} \\
&\leq (q-1)\{(k-1)x_1 + x_{n-k+2}\} + (k-1)x_1 + x_{n-k+2} \\
&= q\{(k-1)x_1 + x_{n-k+2}\} \leq 1.
\end{aligned}$$

Here equality holds only if at least one of (a). $q = k-1$ and $x_{n-k+2} = x_n$ or (b). $x_1 = x_{n-k+2} = x_n$ is satisfied. If neither of these holds, then we get the contradiction $\sum_{i=1}^{n} x_i < 1$. Assume (a) to be true. Then we obtain $x_1 > y_1$ and $x_{n-k+2} = x_n > y_n$ which contradicts the definition of K_2. If (b) is true, then $x_i = 1/n$ for all i and thus x cannot dominate y.

<u>External stability</u>: Take any $x \in A - (K_1 \cup K_2)$.

<u>Case (i)</u> $x_{n-k+1} \geq 1/q(k-1)$: Let $(n-1)\varepsilon = \sum_{i=1}^{n} x_i - (n-k+1)x_{n-k+1} - (k-2)x_{n-1}$. Then $\varepsilon > 0$ since $x \notin K_1$. Define y by

$$y_i = \begin{cases} x_{n-k+1} + \varepsilon & \text{for } i = 1,\dots,n-k+1 \\ x_{n-1} + \varepsilon & \text{for } i = n-k+2,\dots,n-1 \\ 0 & \text{for } i = n. \end{cases}$$

Then $y \in K_1$. Now we will show that $\{1,\dots,k-2,n-k+2,n-k+3\}_y$ is effective for y. If this is true, then we obtain $y \operatorname{dom} x$ via $\{1,\dots,k-2,n-k+2,n-k+3\}_y \,|\, \{n-k+1,\dots,n\}_x$. Let us assume $(k-2)x_1 + 2x_{n-k+2} > 1/q$. First assume k to be even, say $k = 2\ell$, then $q \geq [[(k+1)/2]] = \ell$. Thus we get

$$\begin{aligned}\sum_{i=1}^{n} y_i &= (n-k+1)y_1 + (k-2)y_{n-k+2} \\ &= (\ell-1)((2\ell-2)y_1 + 2y_{n-k+2}) + (2q\ell-2\ell^2+2\ell-1)y_1 \\ &> (\ell-1)/q + (2q\ell-2\ell^2+2\ell-1)/q(2\ell-1) \\ &= 1 + (q-\ell)/q(2\ell-1) \geq 1.\end{aligned}$$

Now assume k to be odd, say $k = 2\ell + 1$, then $q \geq \ell + 1$. Hence

$$\begin{aligned}\sum_{i=1}^{n} y_i &= (\ell-1)((2\ell-1)y_1 + 2y_{n-k+2}) + (2q\ell+q+\ell-2\ell^2-1)y_1 + y_{n-k+2} \\ &> (\ell-1)/q + (2q\ell+q+\ell-2\ell^2-1)/2q\ell \\ &= 1 + (q-(\ell+1))/2q\ell \geq 1.\end{aligned}$$

Thus we reach contradictions for both cases. Therefore $\{1,\ldots,k-2, n-k+2, n-k+3\}_y$ is effective for y.

<u>Case (ii)</u> $x_{n-k+1} < 1/q(k-1)$: Let $\varepsilon = \sum_{i=1}^{n} x_i - (n-k+1)x_{n-k+1} - (k-2)x_{n-1} - x_n$.

<u>(ii-I)</u> $\varepsilon > 0$: Define y by

$$y_i = \begin{cases} x_{n-k+1} + \varepsilon' & \text{for } i = 1,\ldots,n-k+1 \\ x_{n-1} + \varepsilon'' & \text{for } i = n-k+2,\ldots,n-1 \\ x_n + \varepsilon''' & \text{for } i = n \end{cases}$$

where $\varepsilon', \varepsilon'', \varepsilon''' > 0$, $(n-k+1)\varepsilon' + (k-2)\varepsilon'' + \varepsilon''' = \varepsilon$ and $1/q(k-1) > y_1$.

If $y \in K_2$, then y dom x via $\{1,\ldots,k-1,n\}_y | \{n-k+1,\ldots,n\}_x$ since $\{1,\ldots,k-1,n\}_y$ is effective for y. Now assume $y \notin K_2$. Then there must exist some $z \in K_2$ such that (a). $z_1 > y_1$ and $z_{n-k+2} > y_{n-k+2}$ or (b). $z_1 > y_1$ and $z_n > y_n$ from the definition of K_2. In either case, we obtain z dom x.

<u>(ii-II)</u> $\varepsilon = 0$: We can assert $x \in \text{Dom } K_2$ in a manner similar to that above. $\square$

As an analogue of Theorem 6.3, we can obtain the next theorem which gives us a symmetric stable set for the case where $r \geq 1$.

<u>Theorem 6.4</u>: Define

$K_1 = \langle \{x \in \lceil A \rceil \mid x_1 = \ldots = x_{n-k+1} \geq 1/q(k-1) \geq x_{n-k+2} = \ldots = x_{n-1} \geq x_n = 0\} \rangle,$

$K_2 = \langle \{x \in \lceil A \rceil \mid 1/q(k-1) \geq x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_{n-1} \geq x_n;$

x is on a curve connecting

$$(\underbrace{1/q(k-1),\ldots,1/q(k-1)}_{n-k+1}, \underbrace{(k-q-r-1)/q(k-1)(k-2),\ldots,(k-q-r-1)/q(k-1)(k-2)}_{k-2},0)$$

with $\underbrace{((k-q-1)/q(k^2-qk-k-r),\ldots,(k-q-1)/q(k^2-qk-k-r)}_{n-k+1}$,

$$\underbrace{(k-q-r-1)/q(k^2-qk-k-r),\ldots,(k-q-r-1)/q(k^2-qk-k-r))}_{k-1}$$

where all coordinates $x_1,\ldots,x_n$ vary monotonically;

$(k-1)x_1 + x_n \leq 1/q \} \rangle$

and

$K_3 = \langle \{x \in \lceil A \rceil \mid (k-q-1)/q(k^2-qk-k-r) \geq x_1 = \ldots = x_{n-k+1} - 1/qk \geq x_{n-k+2} = \ldots = x_n;$
$(k-1)x_1 + x_{n-k+2} \leq 1/q\} \rangle.$

Then $\bigcup_{i=1}^{3} K_i$ is a stable set for $(n;k)_h$ games with $r \geq 1$ if $[[(k-r)/2]] \leq q \leq k-(r+2)$ where $[[(k-r)/2]]$ is the greatest integer in $(k-r)/2$.

<u>Proof</u>: We will only prove the following two properties about effectiveness, since the other parts of the proof proceed similarly to Theorem 6.3.

(a). If $x \in K_2$, then $\{1,\ldots,k-1, n-k+2\}_x$ is effective for x only when $x_1 = (k-q+1)/q(k^2-qk-k-r)$ and $x_{n-k+2} = (k-q-r-1)/q(k^2-qk-k-r)$. In fact, if we assume $(k-1)x_1 + x_{n-k+2} \leq 1/q$, then

$$\sum_{i=1}^{n} x_i = (qk+r-k+1)x_1 + (k-2)x_{n-k+2} + x_n$$

$$= (qk+r-k+1)(x_1+x_{n-k+2}/(k-1)) + (k-2-(qk+r-k+1)/(k-1))x_{n-k+2} + x_n$$

$$\leq (qk+r-k+1)(x_1+x_{n-k+2}/(k-1)) + x_{n-k+2} \cdot (k^2-k-qk-r)/(k-1)$$

$$\leq (qk+r-k+1)/q(k-1) + (k-q-r-1)/q(k-1) = 1$$

where equality holds only if $x_1 = (k-q-1)/q(k^2-qk-k-r)$ and $x_{n-k+2} = (k-q-r-1)/q(k^2-qk-k-r)$.

(b). For all $x \in A$, $\{1,\dots,k-2, n-k+2, n-k+3\}_x$ is effective. Assume $(k-2)x_1 + 2x_{n-k+2} > 1/q$. Then

$$\sum_{i=1}^{n} x_i = (qk+r-k+1)x_1 + (k-2)x_{n-k+2} + x_n \geq q(k-2)x_1 + (r+2q-1)x_{n-k+2}$$

$$\geq q(k-2)x_1 + 2qx_{n-k+2} > 1$$

since $[[(k-r)/2]] \leq q$ and $r \geq 1$. □

Now let us digress in our discussion and consider what we have obtained so far. For the sake of convenience, we will summarize our result in Table 6.1. In this table, games below the fine lines have K_h as their symmetric stable sets. In particular, for any game on the righthand side of the bold lines, K_h is unique. Games marked by "_" are those games whose symmetric stable sets have been initially described in Theorems 6.3 and 6.4. Here attention must be paid to the fact that $(17;7)_h$ is marked by "_", i.e. $(17;7)_h$ has the symmetric stable set defined in Theorem 6.4.

$$n = qk + r \quad (q \geq 2, \quad 0 \leq r \leq k-1)$$

k = 2:

q	n	
2 ;	4	5
3 ;	6	7
...	...	

k = 3:

q	n		
2 ;	6	7	8
3 ;	9	10	11
...	...		

k = 4:

q	n			
2 ;	$\underline{8}$	9	10	11
3 ;	$\underline{12}$	13	14	15
4 ;	16	17	18	19
...	...			

k = 5:

q	n				
2 ;	$\overline{10}$	$\underline{11}$	12	13	14
3 ;	$\underline{15}$	16	17	18	19
4 ;	$\underline{20}$	21	22	23	24
5 ;	25	26	27	28	29
...	...		...		

k = 6:

q	n					
2 ;	$\overline{12}$	$\underline{13}$	$\underline{14}$	15	16	17
3 ;	$\underline{18}$	$\underline{19}$	20	21	22	23
4 ;	$\underline{24}$	25	26	27	28	29
5 ;	$\underline{30}$	31	32	33	34	35
6 ;	$\underline{36}$	37	38	39	40	46
...	...			...		

k = 7:

q	n						
2 ;	$\overline{14}$	15	$\underline{16}$	$\underline{17}$	18	19	20
3 ;	$\overline{21}$	$\underline{22}$	$\underline{23}$	24	25	26	27
4 ;	$\underline{28}$	$\underline{29}$	30	31	32	33	34
5 ;	$\underline{35}$	36	37	38	39	40	41
6 ;	$\underline{42}$	43	44	45	46	47	48
7 ;	49	50	51	52	53	54	55
...	...				...		

k = 8:

q	n							
2 ;	$\overline{16}$	17	18	$\underline{19}$	$\underline{20}$	21	22	23
3 ;	24	$\underline{25}$	$\underline{26}$	$\underline{27}$	28	29	30	31
4 ;	$\underline{32}$	$\underline{33}$	$\underline{34}$	35	36	37	38	39
5 ;	$\underline{40}$	$\underline{41}$	42	43	44	45	46	47
6 ;	$\underline{48}$	49	50	51	52	53	54	55
7 ;	$\underline{56}$	57	58	59	60	61	62	63
8 ;	64	65	66	67	68	69	70	71
...	...					...		

Table 6.1 Hart games $(n;k)_h$

For this $(17;7)_h$ game, Hart also gives a symmetric stable set of another type, namely

$$K = K_1 \cup K_2$$

where

$$K_1 = \langle\{x \in \lceil A\rceil \mid 1/11 \geq x_1 = \dots = x_{n-k+1} \geq 3/38 \geq x_{n-k+2} = \dots = x_{n-1} \geq x_n = 0\}\rangle,$$

and

$$K_2 = \langle\{x \in \lceil A\rceil \mid 16/209 \geq x_1 = \dots = x_{n-k+1} \geq 1/14 \geq x_{n-k+2} = \dots = x_n\}\rangle.$$

Here it follows from our Theorem 6.4 that his claim of the uniqueness of his stable set is false. The next theorem shows that all games satisfying the condition in Theorem 6.4 have symmetric stable sets of this type.

<u>Theorem 6.5</u>: Define

$$K_1 = \langle\{x \in [A] \mid x_1 = \dots = x_{n-k+1} \geq x_{n-k+2} = \dots = x_{n-1} \geq x_n = 0;$$
$$(k-1)x_1 + x_{n-k+2} \geq 1/q\}\rangle$$

and

$$K_2 = \langle\{x \in [A] \mid x_1 = \dots = x_{n-k+1} \geq x_{n-k+2} = \dots = x_n;$$

if $(q^2k-qk^2+qk+qr-rk+r-2k+1+k^2)/q(qk-k^2+2k-1+r)\cdot(qk+r-k+1) \geq 1/qk$, then $(q^2k-qk^2+qk+qr-rk+r-2k+1+k^2)/q(qk-k^2+2k-1+r)(qk+r-k+1) \geq x_1 \geq 1/qk$;

otherwise $x_1 = (q^2k-qk^2+qk+qr-rk+r-2k+1+k^2)/q(qk-k^2+2k-1+r)(qk+r-k+1)\}\rangle$.

Then $K_1 \cup K_2$ is a symmetric stable set if $[[(k-r)/2]] \leq q \leq k-(r+2)$.

Proof: We will only point out the following properties since the rest of this proof is similar to Theorem 6.3.

(a). If $x \in K_1$, then $x_1 \geq (q+2-k)/q(qk-k^2+2k-1+r)$ and $x_{n-k+2} \leq (q+r+1-k)/q(qk-k^2+2k-1+r)$.

(b). If $x \in K_2$, then $x_{n-k+2} \geq (q+r+1-k)/q(qk-k^2+2k-1+r)$.

These two properties are easily verified by simple calculations. □

Now let us return to Table 6.1. This table tells us that for all games $(n;k)_h$ with $k \leq 4$, symmetric stable sets have been obtained. For $k = 5$ and 6, only $(10;5)_h$ and $(12;6)_h$ are unsolved to this point. Thus we next concentrate on these games and determine their symmetric stable sets.

Theorem 6.6: Assume $n = qk$, $k = 2\ell+1$ $(\ell \geq 2)$ and $q = \ell$. ($(10;5)_h$, $(21;7)_h$, $(36;9)_h$, etc. satisfy these conditions.) Define

$$K_1 = \langle \{x \in \lceil A \rceil \mid x_1 = \dots = x_{n-k+1} \geq 1/q(k-1) \geq x_{n-k+2} = \dots = x_{n-1} \geq x_n = 0\} \rangle,$$

$$K_2 = \langle \{x \in \lceil A \rceil \mid x_1 = \dots = x_{n-k+1} \geq 1/q(k-1) \geq x_{n-k+2} = \dots = x_{n-2} \geq x_{n-1} = x_n;$$
$$(n-k+1)x_1 + (k-2)x_{n-k+2} = 1\} \rangle,$$

$$K_3 = \langle \{x \in \lceil A \rceil \mid 1/q(k-1) \geq x_1 = \dots = x_{n-k+1} \geq x_{n-k+2} = \dots = x_{n-1} \geq x_n;$$
$$(n-2k+2)x_1 + (k-2)x_{n-k+2} = (q-1)/q\} \rangle$$

and

$K_4 = \langle\{x \in \lceil A\rceil \,|\, 1/q(k-1) \geq x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_{n-2} \geq x_{n-1} = x_n;$

$$(n-2k+2)x_1 + (k-2)x_{n-k+2} = (q-1)/q\}\rangle.$$

Then $\bigcup_{i=1}^{4} K_i$ is a symmetric stable set.

<u>Proof</u>: <u>Internal stability</u>: Take any $x,y \in \bigcup_{i=1}^{4} K_i$ and assume $x \text{ dom } y$ via $S_x|\{n-k+1,\ldots,n\}_y$. Let $S_x^1 = S_x \cap \{1,\ldots,n-k+1\}_x$, $S_x^2 = S_x \cap \{n-k+2,\ldots,n-2\}_x$ and $S_x^3 = S_x \cap \{n-k+2,\ldots,n-1\}_x$.

<u>Case (i)</u> $x \in K_1$: We first note that $|S_x^1| \leq k-3$. In fact, if $|S_x^1| \geq k-1$, then $\sum_{i \in S_x} x_i > 1/q$. Suppose $|S_x^1| = k-2$ and S_x is effective for x. Then we get the contradiction

$$\sum_{i=1}^{n} x_i = (n-k+1)x_1 + (k-2)x_{n-k+2} = \ell(2\ell-1)x_1 + (2\ell-1)x_{n-k+2}$$

$$\leq \ell(2\ell-1)x_1 + 2\ell x_{n-k+2} \leq 1$$

where equality holds only if $x_{n-k+2} = 0$.

Therefore, if $y \in K_1 \cup K_2$, then we get the contradiction $1 = (n-k+1)x_1 + (k-2)x_{n-k+2} > (n-k+1)y_1 + (k-2)y_{n-k+2} = 1$. For $y \in K_3 \cup K_4$, $x \text{ d}\not{o}\text{m } y$ since $x_{n-k+2} < y_{n-k+2}$.

<u>Case (ii)</u> $x \in K_2$: We must have $|S_x^1| \leq k-2$ since $x_1 > 1/q(k-1)$. Moreover if $|S_x^1| = k-2$, then $|S_x^2|$ should be less than or equal to 1. In fact, if $|S_x^2| = 2$ and S_x is effective, then

$$\sum_{i=1}^{n} x_i = (n-k+1)x_1 + (k-3)x_{n-k+2} + 2x_{n-1} = \ell(2\ell-1)x_1 + (2\ell-2)x_{n-k+2} + 2x_{n-1}$$

$$\leq \ell(2\ell-1)x_1 + 2\ell x_{n-k+2} \leq 1.$$

Therefore, for $y \in K_1$ we get the contradiction $1 = (n-k+1)x_1 + (k-2)x_{n-k+2} > (n-k+1)y_1 + (k-2)y_{n-k+2} = 1$. For $y \in K_2$, if $|S_x^1| = k-2$, then we get $x_{n-k+2} > y_{n-k+2}$ and thus the same contradiction is deduced. If $|S_x^1| < k-2$, then clearly we get $x_{n-k+2} > y_{n-k+2}$. For $y \in K_3 \cup K_4$, x døm y.

<u>Case (iii)</u> $x \in K_3$: $|S_x^1|$ must be less than or equal to $k-1$. Moreover if $|S_x^1| = k-1$, then $|S_x^3| = 0$ and if $|S_x^1| = k-2$, then $|S_x^3| = 1$. Therefore we get the contradiction $1/q = (k-1)x_1 + x_n > (k-1)y_1 + y_n \geq 1/q$ or $(q-1)/q = (n-2k+2)x_1 + (k-2)x_{n-k+2} > (n-2k+2)y_1 + (k-2)y_{n-k+2} = (q-1)/q$ for $y \in K_3 \cup K_4$. For $y \in K_1 \cup K_2$, x døm y since $x_1 \leq 1/q(k-1) \leq y_1$.

<u>Case (iv)</u> $x \in K_4$: S_x must have $(n-1)_x$ or n_x among its members and furthermore $\{1,\ldots,k-1, n-1\}_x$ is not effective. Hence we get the contradiction for $y \in K_3 \cup K_4$. For $y \in K_1 \cup K_2$, x døm y.

<u>External stability</u>: Take any $x \in A - \bigcup_{i=1}^{4} K_i$. Let $n\varepsilon = \sum_{i=1}^{n-2} x_i - (n-k+1)x_{n-k+1} - (k-3)x_{n-2}$ and define y by

$$y_i = \begin{cases} x_{n-k+1} + \varepsilon & \text{for } i = 1,\ldots,n-k+1 \\ x_{n-2} + \varepsilon & \text{for } i = n-k+2,\ldots,n-2 \\ x_i + \varepsilon & \text{for } i = n-1,n. \end{cases}$$

<u>Case (i)</u> $y_1 \geq 1/q(k-1)$: <u>(i-I)</u> $\varepsilon > 0$: First we consider the case $(n-k+1)y_1 + (k-2)y_{n-k+2} < 1$. Define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\dots,n-k+1 \\ ((k-3)y_{n-k+2} + 2y_{n-1})/(k-2) & \text{for } i = n-k+2,\dots,n-1 \\ 0 & \text{for } i = n. \end{cases}$$

Then $y' \in K_1$ and $y' \operatorname{dom} x$ via $\{1,\dots,k-3,n-k+2,n-k+3,n-k+4\}_{y'}|\{n-k+1,\dots,n\}_x$. Here the effectiveness of $\{1,\dots,k-3, n-k+2, n-k+3, n-k+4\}_{y'}$ follows from $y'_1 > 1/q(k-1)$. In fact, if we assume that this is not effective, then we get the contradiction

$$\sum_{i=1}^{n} y_i = (n-k+1)y'_1 + (k-2)y'_{n-k+2} = \ell(2\ell-1)y'_1 + (2\ell-1)y'_{n-k+2}$$

$$\geq 2\ell y'_1 + (\ell-1)((2\ell-2)y'_1 + 3y'_{n-k+2}) > 1/\ell + (\ell-1)/\ell = 1.$$

Furthermore $y'_{n-k+2} > y_{n-k+2}$ since $(n-k+1)y_1 + (k-2)y_{n-k+2} < 1$. Thus $y'_1 = \dots = y'_{k-3} = y_1 > x_{n-k+1} \geq x_{n-k+2} \geq \dots \geq x_{n-3}$ and $y'_{n-k+2} = y'_{n-k+3} = y'_{n-k+4} > y_{n-k+2} > x_{n-2} \geq x_{n-1} \geq x_n$.

Now assume $(n-k+1)y_1 + (k-2)y_{n-k+2} \geq 1$. Define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\dots,n-k+1 \\ (1-(n-k+1)y_1)/(k-2) & \text{for } i = n-k+2,\dots,n-2 \\ (1-(n-k+1)y_1)/2(k-2) & \text{for } i = n-1,n. \end{cases}$$

Then $y' \in K_2$ and $y' \operatorname{dom} x$ via $\{1,\dots,k-2, n-k+2, n-1\}_{y'}|\{n-k+1,\dots,n\}_x$.

The effectiveness of $\{1,\ldots,k-2,\ n-k+2,\ n-1\}_{y'}$ is shown as follows. Assume otherwise, then

$$\sum_{i=1}^{n} y_i' = (n-k+1)y_1' + (k-3)y_{n-k+2}' + 2y_{n-1}' = \ell(2\ell-1)y_1' + (2\ell-2)y_{n-k+2}' + 2y_{n-1}'$$

$$= \ell\{(2\ell-1)y_1' + y_{n-k+2}' + y_{n-1}'\} + (\ell-2)(y_{n-k+2}' - y_{n-1}') > 1.$$

Now we will show $y_{n-k+2}' \geq y_{n-1}$ and $y_{n-1}' \geq y_n$. First, assume $y_{n-k+2}' < y_{n-1}$, then $(n-k+1)y_1 + (k-2)y_{n-1} > 1$. Thus we get the contradiction

$$\sum_{i=1}^{n} y_i = (n-k+1)y_1 + (k-3)y_{n-k+2} + y_{n-1} + y_n \geq (n-k+1)y_1 + (k-2)y_{n-1} > 1.$$

Second, assume $y_{n-1}' < y_n$, then $(n-k+1)y_1 + 2(k-2)y_n > 1$. Together with $y_{n-k+2} \geq y_{n-1} + y_n$, we have the contradiction

$$\sum_{i=1}^{n} y_i = (n-k+1)y_1 + (k-3)y_{n-k+2} + y_{n-1} + y_n \geq (n-k+1)y_1 + (k-2)(y_{n-1} + y_n)$$

$$\geq (n-k+1)y_1 + 2(k-2)y_n > 1.$$

Therefore $y_1' = \ldots = y_{k-2}' = y_{n-k+1}' > x_{n-k+1} \geq \ldots \geq x_{n-2}$, $y_{n-k+2}' \geq y_{n-1} > x_{n-1}$ and $y_{n-1}' \geq y_n > x_n$.

<u>(i-II)</u> $\varepsilon = 0$: In this case $y_i = x_i$ for $i = 1,\ldots,n$. We first assume $(n-k+1)y_1 + (k-2)y_{n-k+2} < 1$, then we must have $y_n > 0$. Define y' by

$$y_i' = \begin{cases} y_i + \varepsilon' & \text{for } i = 1,\ldots,n-1 \\ y_n - (n-1)\varepsilon' & \text{for } i = n \ \ldots \end{cases}$$

where $\varepsilon' > 0$ is sufficiently small so that $(n-k+1)y_1' + (k-2)y_{n-k+2}' < 1$ and $y_n' > 0$.

When $(n-k+1)y_1 + (k-2)y_{n-k+2} > 1$ holds, we must have $y_{n-3} > y_{n-2}$. Define y' by

$$y_i' = \begin{cases} y_i + \varepsilon' & \text{for } i = 1,\ldots,n-k+1 \\ y_i - \varepsilon'' & \text{for } i = n-k+2,\ldots,n-2 \\ y_i + \varepsilon''' & \text{for } i = n-1,\ n \end{cases}$$

where $\varepsilon', \varepsilon'', \varepsilon''' > 0$ are sufficiently small so that $y' \in \lceil A \rceil$ and $(n-k+1)y_1' + (k-2)y_{n-k+2}' > 1$.

If $(n-k+1)y_1 + (k-2)y_{n-k+2} = 1$, then $y_{n-1} > y_n > 0$ since $y = x \notin K_1 \cup K_2$. Define y' by

$$y_i' = \begin{cases} y_i + \varepsilon' & \text{for } i = 1,\ldots,n-1 \\ y_n - (n-1)\varepsilon' & \text{for } i = n \end{cases}$$

where $\varepsilon' > 0$ is sufficiently small so that $y_n' > 0$.

Using these y', we can take some $y'' \in K_1 \cup K_2$ which dominates x in a manner similar to Case (i-I).

<u>Case (ii)</u> $y_1 < 1/q(k-1)$: <u>(ii-I)</u> $\varepsilon > 0$: Let us first assume $(n-2k+2)y_1 + (k-2)y_{n-k+2} < (q-1)/q$. Define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\dots,n-k+1 \\ (q-1-q(n-2k+2)y_1)/q(k-2) & \text{for } i = n-k+2,\dots,n-1 \\ (1-q(k-1)y_1)/q & \text{for } i = n. \end{cases}$$

Then $y' \in K$ and y' dom x via $\{1,\dots,k-3,n-k+2,n-k+3,n-k+4\}_{y'}|\{n-k+1,\dots,n\}_x$. The effectiveness of $\{1,\dots,k-3, n-k+2, n-k+3, n-k+4\}_{y'}$ follows from $(n-2k+2)y'_1 + (k-2)y'_{n-k+2} = (q-1)/q$. In fact, if we suppose $(k-3)y'_1 + 3y'_{n-k+2} > 1/q$, then we get

$$(n-2k+2)y'_1 + (k-2)y'_{n-k+2} = (2\ell^2-3\ell)y'_1 + (2\ell-1)y'_{n-k+2}$$

$$\geq (\ell-1)((2\ell-4)y'_1 + 3y'_{n-k+2}) > (\ell-1)/q = (q-1)/q.$$

Now obviously $y'_1 = \dots = y'_{k-3} = y_{n-k+1} > x_{n-k+1} \geq \dots \geq x_{n-3}$. Finally, we obtain

$$y'_{n-k+2} = y'_{n-k+3} = y'_{n-k+4} = (q-1-q(n-2k+2)y_1)/q(k-2)$$

$$= (1/(k-2))((q-1)/q - (n-2k+2)y_1) = y_{n-k+2}.$$

Next assume $(n-2k+2)y_1 + (k-2)y_{n-k+2} \geq (q-1)/q$. Define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\dots,n-k+1 \\ (q-1-q(n-2k+2)y_1)/q(k-2) & \text{for } i = n-k+2,\dots,n-2 \\ (q+k-3-q(k^2-5k+n+4)y_1)/2q(k-2) & \text{for } i = n-1, n. \end{cases}$$

Then it is easy to show that $y' \in K_4$ and $\{1,\dots,k-2,\ n-k+2,\ n-1\}_{y'}$ is effective for y'. Obviously $y'_1 = \dots = y'_{k-2} > x_{n-k+1} \geq \dots \geq x_{n-2}$. Hence if $y'_{n-k+2} \geq y_{n-1}$ and $y'_{n-1} \geq y_n$ then y' dom x via $\{1,\dots,k-2,\ n-k+2,\ n-1\}_{y'}|\{n-k+1,\dots,n\}_x$. Suppose $y'_{n-k+2} < y_{n-1}$, then

$$(k-1)y_1 + y_n = 1 - ((n-2k+2)y_1 + (k-3)y_{n-k+2} + y_{n-1})$$

$$\leq 1 - ((n-2k+2)y_1 + (k-2)y_{n-1}) < 1-(q-1)/q = 1/q$$

which implies that $(1-q(k-1)y_1)/q > y_n$. Therefore the y' defined at the beginning of Case (ii) dominates x via $\{1,\dots,k-1,n\}_{y'}|\{n-k+1,\dots,n\}_x$. Now suppose $y'_{n-1} < y_n$ then we obtain (a). $\ell(6\ell^2-5\ell)y_1 + 2\ell(\ell-1)y_n > 3\ell-2$. From $(n-2k+2)y_1 + (k-2)y_{n-k+2} \geq (q-1)/q$, we get (b). $\ell(2\ell^2-3\ell)y_1 + \ell(2\ell-1)y_{n-k+2} \geq \ell-1$. Add (a) $\times\ 2(\ell-1)$ to (b) to obtain the contradiction $\sum_{i=1}^{n} y_i = \ell(2\ell-1)y_1 + 2(\ell-1)y_{n-k+2} + 2y_n > 1$.

<u>(ii-II)</u> $\varepsilon = 0$: Here we have $y_i = x_i$ for $i = 1,\dots,n$. We first assume $(n-2k+2)y_1 - (k-2)y_{n-k+2} < (q-1)/q$. Then $y_n > 0$ since $y_1 < 1/q(k-1)$. Define y' by

$$y'_i = \begin{cases} y_i + \varepsilon' & \text{for } i = 1,\dots,n-1 \\ y_n - (n-1)\varepsilon' & \text{for } i = n \end{cases}$$

where $\varepsilon > 0$ is sufficiently small so that $y'_1 \leq 1/q(k-1)$, $y'_n > 0$ and $(n-2k+2)y'_1 + (k-2)y'_{n-k+2} < 1/q$.

Second, assume $(n-2k+2)y_1 + (k-2)y_{n-k+2} > (q-1)/q$. If $y_{n-2} > y_{n-1}$, then define y' by

$$y_i' = \begin{cases} y_i + \varepsilon' & \text{for } i = 1,\ldots,n-k+1,n-1,n \\ y_i - \varepsilon'' & \text{for } i = n-k+2,\ldots,n-2 \end{cases}$$

where $\varepsilon', \varepsilon'' > 0$ and $(k-3)\varepsilon'' = (n-k+3)\varepsilon'$ and $\varepsilon', \varepsilon''$ are sufficiently small so that $y_1' \leq 1/q(k-1)$, $y_{n-2}' > y_{n-1}'$ and $(n-2k+2)y_1' + (k-2)y_{n-k+2}' > (q-1)/q$. If $y_{n-2} = y_{n-1}$, then $y_{n-1} > y_n$ since $(n-2k+2)y_1 + (k-2)y_{n-k+2} > (q-1)/q$. Define y' by

$$y_i' = \begin{cases} y_i + \varepsilon' & \text{for } i = 1,\ldots,n-k+1,n \\ y_i - \varepsilon'' & \text{for } i = n-k+2,\ldots,n-2,n-1 \end{cases}$$

where $\varepsilon', \varepsilon'' > 0$ and $(k-2)\varepsilon'' = (n-k+2)\varepsilon'$ and $\varepsilon', \varepsilon''$ are sufficiently small so that $y_1' \leq 1/q(k-1)$, $y_{n-1}' > y_n'$ and $(n-2k+2)y_1' + (k-2)y_{n-k+2}' > (q-1)/q$.

Finally, let us assume $(n-2k+2)y_1 + (k-2)y_{n-k+2} = (q-1)/q$. Then we must have $y_{n-2} > y_{n-1} > y_n$ since $y \notin K_3 \cup K_4$. Define y' by

$$y_i' = \begin{cases} y_i + \varepsilon' & \text{for } i \neq n-1 \\ y_i - (n-1)\varepsilon' & \text{for } i = n-1 \end{cases}$$

where $\varepsilon' > 0$ is sufficiently so that $y'_1 \leq 1/q(k-1)$ and $y'_{n-1} > y'_n$.

By using these y', the proof is similar to that in (ii-I). $\square$

The next theorem will give us a symmetric stable set for $(12;6)_h$.

<u>Theorem 6.7</u>: Define

$$K_1 = \langle\{x \in \lceil A\rceil \mid x_1 = \dots = x_7 \geq 1/8 \geq x_8 = x_9 = x_{10} \geq x_{11} = x_{12} = 0\}\rangle,$$

$$K_2 = \langle\{x \in \lceil A\rceil \mid 1/8 \geq x_1 = \dots = x_7 \geq 1/9 \geq x_8 = x_9 = x_{10} \geq x_{11} \geq x_{12};$$
$$3x_1 + 3x_8 = 1/2\}\rangle,$$

$$K_3 = \langle\{x \in \lceil A\rceil \mid 1/9 \geq x_1 = \dots = x_7 \geq 1/10 \geq x_8 = x_9 = x_{10} = x_{11} \geq x_{12} = 0\}\rangle,$$

$$K_4 = \langle\{x \in \lceil A\rceil \mid 1/9 \geq x_1 = \dots = x_7 \geq 1/10 \geq x_8 = x_9 = x_{10} \geq x_{11} \geq x_{12};$$
$$7x_1 + 4x_8 = 1;\ \ 4x_1 + x_8 + x_{12} \geq 1/2\}\rangle,$$

$$K_5 = \langle\{x \in \lceil A\rceil \mid 1/10 \geq x_1 = \dots = x_7 \geq x_8 = x_9 = x_{10} = x_{11} \geq x_{12};\ 2x_1 + 4x_8 = 1/2\}\rangle,$$

$$K_6 = \langle\{x \in \lceil A\rceil \mid 1/10 \geq x_1 = \dots = x_7 \geq 7/72 \geq x_8 = x_9 = x_{10} \geq x_{11} \geq x_{12};$$
$$2x_1 + 4x_8 = 1/2;\ \ 4x_1 + x_8 + x_{12} \geq 1/2\}\rangle,$$

$$K_7 = \langle\{x \in \lceil A\rceil \mid 7/72 \geq x_1 = \dots = x_7 \geq x_8 = x_9 = x_{10} \geq x_{11} \geq x_{12};$$
$$2x_1 + 4x_8 = 1/2;\ \ 4x_1 + x_8 + x_{12} = 1/2\}\rangle$$

and

$$K_8 = \langle\{x \in \lceil A \rceil \mid 7/72 \geq x_1 = \dots = x_7 \geq x_8 = x_9 \geq x_{10} = x_{11} = x_{12};\ 2x_1 + 4x_8 = 1/2\}\rangle.$$

Then $\bigcup_{i=1}^{8} K_i$ is a symmetric stable set for $(12;6)_h$.

__Proof__: __Internal stability__: Take any $x,y \in \bigcup_{i=1}^{8} K_i$ and assume $x \text{ dom } y$ via $S_x|\{7,8,9,10,11,12\}_y$.

__Case (i)__ $x \in K_1$: S_x must include $\{8,9,10\}_x$ since $x_1 \geq 1/8$ and $x_{11} = x_{12} = 0$. Hence we have the contradiction $\sum_{i=1}^{12} x_i > \sum_{i=1}^{12} y_i$ for $y \in K_1$. For $y \in \bigcup_{i=2}^{7} K_i$, $x \not\text{dom } y$ since $x_8 \leq 1/24 \leq y_8$. For $y \in K_8$, $x \not\text{dom } y$ since $x_8 \leq 1/24 \leq y_{10}$.

__Case (ii)__ $x \in K_2$: Since $x_1 \geq 1/9$, S_x cannot contain more than four elements of $\{1,\dots,7\}_x$. For $y \in K_1$, clearly $x \not\text{dom } y$. For $y \in K_2$, S_x must be of the form $\{4,5,6,7,11,12\}_x$, $\{4,5,6,7,10,12\}_x$, $\{4,5,6,7,10,11\}_x$ or $\{4,5,6,7,9,10\}_x$. We will consider the case where $S_x = \{4,5,6,7,11,12\}_x$. For other cases, the proof will proceed similarly by using the condition $3x_1 + 3x_8 = 1/2$ or $4x_1 + x_{11} + x_{12} = 1/2$. If $S_x = \{4,5,6,7,11,12\}_x$, then we must have $x_1 > y_1$, $x_{11} > y_{11}$ and $x_{12} > y_{12}$. Thus we get the contradiction

$$\sum_{i=1}^{12} x_i = 7x_1 + 3x_8 + x_{11} + x_{12} = 3x_1 + 3x_8 + 4x_1 + x_{11} + x_{12}$$

$$> 3y_1 + 3y_8 + 4y_1 + y_{11} + y_{12} = \sum_{i=1}^{12} y_i$$

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since $3x_1 + 3x_8 = 3y_1 + 3y_8 = 1/2$. For $y \in K_3 \cup K_5$. x dǿm y since $x_8 \leq 1/18 \leq y_8$. For $y \in K_4$, without loss of generality, we can assume $S_x = \{4,5,6,7,11,12\}_x$ since $x_8 \leq 1/18 \leq y_8$. Hence we get the contradiction

$$\sum_{i=1}^{12} x_i = 7x_1 + 3x_8 + x_{11} + x_{12} = (28x_1 + 12x_8 + 4x_{11} + 4x_{12})/4$$

$$> (3(7y_1 + 4y_8) + 7y_1 + 4y_{11} + 4y_{12})/4 = \sum_{i=1}^{12} y_i .$$

The inequality follows from $x_1 > y_1$, $x_{11} > y_{11}$, $x_{12} > y_{12}$ and $7x_1 + 4x_8 \geq 1 = 7y_1 + 4y_8$. For $y \in K_6 \cup K_7$, without loss of generality, we can assume $S_x = \{4,5,6,7,11,12\}_x$. Thus we get the contradiction similar to that above, since $7y_1 + 4y_8 = 2y_1 + 4y_8 + 5y_1 \leq 1 \leq 7x_1 + 4x_8$. Finally for $y \in K_8$, x dǿm y since $x_8 \leq 1/18 \leq y_{10}$.

<u>Case(iii)</u> $x \in K_3$: S_x cannot contain more than three elements of $\{1,\ldots,7\}_x$. For $y \in K_1 \cup K_2$, x dǿm y since $x_1 \leq 1/9 \leq y_1$. For $y \in K_3$, we easily get the contradiction $\sum_{i=1}^{12} x_i > \sum_{i=1}^{12} y_i$. For $y \in K_4 \cup K_5 \cup K_6 \cup K_7$, x dǿm y since $7y_1 + 4y_8 = 1$ (for $y \in K_4$) and $y_8 \geq 3/40$. Finally, assume $y \in K_8$, then S_x must be of the form $\{5,6,7,9,10,11\}_x$. However this S_x is effective only if $x_1 = \ldots = x_7 = 1/9$ and $x_8 = x_9 = x_{10} = x_{11} = 1/18$. And thus x dǿm y since $y_{10} \geq 1/18$.

<u>Case (iv)</u> $x \in K_4$: S_x cannot contain more than four elements of $\{1,\ldots,7\}_x$ since $x_1 \geq 1/10$. For $y \in K_1 \cup K_2$, x dǿm y since $x_1 \leq 1/9 \leq y_1$. For $y \in K_3$, x dǿm y since $7x_1 + 4x_8 = 7y_1 + 4y_8 = 1$.

For $y \in K_4$, without loss of generality, we can assume $S_x = \{4,5,6,7,11,12\}_x$ or $\{4,5,6,7,10,12\}_x$ since $4x_1 + x_8 + x_{12} \geq 1/2$. Let us first assume $S_x = \{4,5,6,7,11,12\}_x$. Then $x_1 > y_1$, $x_{11} > y_{11}$ and $x_{12} > y_{12}$. Hence

$$\sum_{i=1}^{12} x_i = 7x_1 + 3x_8 + x_{11} + x_{12} = (21x_1 + 12x_8 + 7x_1 + 4x_{11} + 4x_{12})/4$$

$$> (21y_1 + 12y_8 + 7y_1 + 4y_{11} + 4y_{12})/4 = \sum_{i=1}^{12} y_i$$

since $7x_1 + 4x_8 = 7y_1 + 4y_8 = 1$. If $S_x = \{4,5,6,7,10,12\}_x$, then $x_1 > y_1$ and $x_{12} > y_{12}$. Moreover from the effectiveness of S_x, $4x_1 + x_8 + x_{12} = 1/2$ and thus $3x_1 + 2x_8 + x_{11} = 1/2 \geq 3y_1 + 2y_8 + y_{11}$. Hence we get the contradiction

$$\sum_{i=1}^{12} x_i = 7x_1 + 3x_8 + x_{11} + x_{12} \geq (28x_1 + 12x_8 + 4x_{11} + 4x_{12})/4$$

$$= (12x_1 + 8x_8 + 4x_{11} + 7x_1 + 4x_8 + 9x_1 + 4x_{12})/4$$

$$> (12y_1 + 8y_8 + 4y_{11} + 7y_1 + 4y_8 + 9y_1 + 4y_{12})/4 = \sum_{i=1}^{12} y_i .$$

For $y \in K_5$, x døm y since $y_8 \geq 3/40 \geq x_8$. For $y \in K_6 \cup K_7$, we can assume $S_x = \{4,5,6,7,11,12\}_x$ or $\{4,5,6,7,10,12\}_x$ since $y_8 \geq 3/40$. For either case, the contradiction is deduced similar to that in the case where $y \in K_4$, since $7y_1 + 4y_8 \leq 1$ and $4y_1 + y_8 + y_{12} \geq 1/2$. Finally, assume $y \in K_8$. First, we note that $x_{11} \leq 1/18$. In fact, if we assume $x_{11} > 1/18$, then we get the contradiction

$$1 = \sum_{i=1}^{12} x_i = 7x_1 + 3x_8 + x_{11} + x_{12} = (28x_1 + 12x_8 + 4x_{11} + 4x_{12})/4$$

$$= (7(4x_1 + x_8 + x_{12}) + 5(x_{11} + x_{12}) + 4x_{11} - 3x_{12})/4$$

$$= (7(4x_1 + x_8 + x_{12}) + 9x_{11} + 2x_{12})/4 > 1$$

since $4x_1 + x_8 + x_{12} \geq 1/2$ and $7x_1 + 4x_8 = 1$. Together with the fact that $y_{10} \geq 1/18$, S_x must be of the form $\{5,6,7,8,9,10\}_x$. However this S_x is effective only when $x_1 = 1/9$, $x_8 = 1/18$ and $x_{11} + x_{12} = 1/18$. Hence x d$\not{o}$m y.

Case (v) $x \in K_5$: For $y \in \bigcup_{i=1}^{4} K_i$, obviously x d$\not{o}$m y. For $y \in K_5$, we easily obtain the contradiction since $2x_1 + 4x_8 = 1/2 = 2y_1 + 4y_8$. For $y \in K_6 \cup K_7$, S_x must be of the form $\{4,5,6,7,11,12\}_x$. Hence $x_1 > y_1$ and $x_{12} > y_{12}$ and thus

$$\sum_{i=1}^{12} x_i = 7x_1 + 4x_8 + x_{12} = 5x_1 + x_{12} + 2x_1 + 4x_8$$

$$> 5y_1 + y_{12} + 2y_1 + 4y_8 \geq \sum_{i=1}^{12} y_i$$

since $2x_1 + 4x_8 = 1/2 = 2y_1 + 4y_8$. Finally, for $y \in K_8$, we obtain $x_1 > y_1$ and $x_{10} = x_{11} \geq x_{12} > y_{10} = y_{11} = y_{12}$, since S_x must contain 12_x. Therefore we get

$$\sum_{i=1}^{12} x_i = 7x_1 + 4x_8 + x_{12} = 6x_1 + x_{10} + x_{11} + x_{12} + x_1 + 2x_8$$

$$> 6y_1 + y_{10} + y_{11} + y_{12} + y_1 + 2y_8 = \sum_{i=1}^{12} y_i$$

since $2x_1 + 4x_8 = 1/2 = 2y_1 + 4y_8$.

Case (vi) $x \in K_6 \cup K_7$: For $y \in \bigcup_{i=1}^{4} K_i$, x døm y is evident. For $y \in K_5$, x døm y since $4x_1 + x_8 + x_{12} \geq 1/2$. For $y \in K_6 \cup K_7$, S_x must be of the form $\{4,5,6,7,10,12\}_x$ since $4x_1 + x_8 + x_{12} \geq 1/2$. Thus we have $x_1 > y_1$, $x_8 > y_{11}$ and $x_{12} > y_{12}$. Hence

$$\sum_{i=1}^{12} x_i = 7x_1 + 3x_8 + x_{11} + x_{12} = (14x_1 + 6x_8 + 2x_{11} + 2x_{12})/2$$

$$= (6x_1 + 4x_8 + 2x_{11} + x_1 + 2x_8 + 7x_1 + 2x_{12})/2$$

$$> (6y_1 + 4y_8 + 2y_{11} + y_1 + 2y_8 + 7y_1 + 2y_{12})/2 = \sum_{i=1}^{12} y_i$$

since $3x_1 + 2x_8 + x_{11} = 1/2 \geq 3y_1 + 2y_8 + y_{11}$ and $x_1 + 2x_8 = 1/4 = y_1 + 2y_8$. For $y \in K_8$, we need to consider only the case where $S_x = \{5,6,7,9,10,11\}_x$. In this case, we obtain $x_1 > y_1$ and $x_{11} > y_{12} = y_{11}$ and thus

$$\sum_{i=1}^{12} x_i = 7x_1 + 3x_8 + x_{11} + x_{12} = x_1 + 2x_8 + 4x_1 + x_8 + x_{12} + 2x_1 + x_{11}$$

$$> 1/4 + 1/2 + 1/4 = 1$$

since $2x_1 + x_{11} > 2y_1 + y_{11} = 1/4$.

Case (vii) $x \in K_8$: For $y \in \bigcup_{i=1}^{6} K_i$, clearly x døm y. Assume $y \in K_7$, then without loss of generality, we can assume $S_x = \{4,5,6,7,11,12\}_x$. Hence $x_1 > y_1$, $x_{11} > y_{11}$ and $x_{12} > y_{12}$, and thus we get the contradiction $1/4 = 2x_1 + x_{11} > 2y_1 + y_{11} = 1/4$ since $y_1 + 2y_8 = 1/4$ and $4y_1 + y_8 + y_{12} = 1/2$. Finally, for $y \in K_8$, evidently x døm y

since $2x_1 + 4x_8 = 1/2 = 2y_1 + 4y_8$ and thus $2x_1 + x_{10} = 1/4 = 2y_1 + y_{10}$.

<u>External stability</u>: Pick any $x \in A - \bigcup_{i=1}^{8} K_i$.

<u>Case (i)</u> $x_7 \geq 1/8$: Let $10\varepsilon = \sum_{i=1}^{12} x_i - (7x_1 + 3x_{10})$ and define y by

$$y_i = \begin{cases} x_7 + \varepsilon & \text{for } i = 1,\ldots,7 \\ x_{10} + \varepsilon & \text{for } i = 8,9,10 \\ 0 & \text{for } i = 11,12. \end{cases}$$

Then trivially $y \in K_1$. If $\varepsilon = 0$, then $x = y \in K_1$. If $\varepsilon > 0$, then y dom x via $\{5,6,7,8,9,10\}_y | \{7,8,9,10,11,12\}_x$ since $y_1 > 1/8$.

<u>Case (ii)</u> $1/8 > x_7 \geq 1/9$: Let $\varepsilon = \sum_{i=1}^{10} x_i - (7x_1 + 3x_{10})$.

<u>(ii-I)</u> $\varepsilon > 0$: Define y by

$$y_i = \begin{cases} x_7 + \varepsilon' & \text{for } i = 1,\ldots,7 \\ x_{10} + \varepsilon'' & \text{for } i = 8,9,10, \\ x_{11} + \varepsilon''' & \text{for } i = 11 \\ x_{12} + \varepsilon^{iv} & \text{for } i = 12 \end{cases}$$

where

$\varepsilon', \varepsilon'', \varepsilon''', \varepsilon^{iv} > 0$, $7\varepsilon' + 3\varepsilon'' + \varepsilon''' + \varepsilon^{iv} = \varepsilon$ and $y_1 = \ldots = y_7 \leq 1/8$.

If $3y_1 + 3y_8 \leq 1/2$, then define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,7 \\ y_i + \delta/3 & \text{for } i = 8,9,10 \\ y_i - \delta' & \text{for } i = 11 \\ y_i - \delta'' & \text{for } i = 12 \end{cases}$$

where $\delta', \delta'' \geq 0$, $\delta' + \delta'' = \delta = 4y_1 + y_{11} + y_{12} - 1/2 > 0$ and $y'_{11} \geq y'_{12}$. Then clearly $y' \in K_2$ and y' dom x via $\{5,6,7,8,9,10\}_{y'} | \{7,8,9,10,11,12\}_x$ since $4y'_1 + y'_{11} + y'_{12} = 1/2$. If $3y_1 + 3y_8 > 1/2$, then define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,7 \\ y_i - \delta/3 & \text{for } i = 8,9,10 \\ y_i + \delta/2 & \text{for } i = 11,12 \end{cases}$$

where $\delta = 3y_1 + 3y_8 - 1/2 > 0$. Here $y'_{10} \geq y'_{11}$ is shown as follows. Assume otherwise. Then $3y'_1 + 3y'_{11} > 3y'_1 + 3y'_8 = 1/2$. Hence $y'_1 + y'_{11} > 1/6$ and thus

$$\sum_{i=1}^{12} y'_i = 7y'_1 + 3y'_8 + y'_{11} + y'_{12} > 7y'_1 + 3y'_8 + y'_{11}$$

$$= y'_1 + y'_{11} + 3y'_1 + 3y'_8 + 3y'_1 > 1/6 + 1/2 + 1/3 = 1.$$

It is easily shown that $y' \in K_2$ and y' dom x via $\{4,5,6,7,11,12\}_{y'} | \{7,8,9,10,11,12\}_x$ since $3y'_1 + 3y'_8 = 1/2$. If

$3y_1 + 3y_8 = 1/2$, then $y \in K_2$ and y dom x via $\{5,6,7,8,9,10\}_y | \{7,8,9,10,11,12\}_x$.

<u>(ii-II)</u> $\varepsilon = 0$: Since $x \notin \bigcup_{i=1}^{8} K_i$, $3x_1 + 3x_8 \neq 1/2$. If $3x_1 + 3x_8 > 1/2$, then define y by

$$y_i = \begin{cases} x_1 + \varepsilon'/9 & \text{for } i = 1,\ldots,7 \\ x_{10} - \varepsilon'/3 & \text{for } i = 8,9,10 \\ x_i + \varepsilon'/9 & \text{for } i = 11,12 \end{cases}$$

where $0 < \varepsilon' < \min(1/8 - x_1,\ 3x_1 + 3x_8 - 1/2)$. Here $y_{10} > y_{11}$ is shown as before. If $3y_1 + 3y_8 < 1/2$, then define y by

$$y_i = \begin{cases} x_1 + \varepsilon'/10 & \text{for } i = 1,\ldots,10 \\ x_{11} - \varepsilon'' & \text{for } i = 11 \\ x_{12} - \varepsilon''' & \text{for } i = 12 \end{cases}$$

where $0 < \varepsilon' < \min(1/8 - x_1,\ 4x_1 + x_{11} + x_{12} - 1/2)$, $\varepsilon'', \varepsilon''' > 0$ and $\varepsilon'' + \varepsilon''' = \varepsilon'$.

Using these y, the proof proceeds in a manner similar to that in (ii-I).

<u>Case (iii)</u> $1/9 > x_7 \geq 1/10$: Let $\varepsilon = \sum_{i=1}^{9} x_i - (7x_1 + 2x_9)$.

<u>(iii-I)</u> $\varepsilon > 0$: Define y by

$$\begin{cases} x_7 + \varepsilon' & \text{for } i = 1,\ldots,7 \\ x_9 + \varepsilon'' & \text{for } i = 8,9 \end{cases}$$

$$y_i = \begin{cases} x_{10} + \varepsilon''' & \text{for } i = 10 \\ x_{11} + \varepsilon^{iv} & \text{for } i = 11 \\ x_{12} + \varepsilon^{v} & \text{for } i = 12 \end{cases}$$

where $\varepsilon', \varepsilon'', \varepsilon''', \varepsilon^{iv}, \varepsilon^{v} > 0$, $7\varepsilon' + 2\varepsilon'' + \varepsilon''' + \varepsilon^{iv} + \varepsilon^{v} = \varepsilon$ and $y_1 \leq 1/9$.

If $7y_1 + 4y_8 < 1$, then define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,7 \\ (2y_8 + y_{10} + y_{11} + y_{12})/4 & \text{for } i = 8,9,10,11 \\ 0 & \text{for } i = 12. \end{cases}$$

Then $y' \in K_3$ and y' dom x via $\{6,7,8,9,10,11\}_{y'} | \{7,8,9,10,11,12\}_x$. In fact, $\{6,7,8,9,10,11\}_{y'}$ is effective since $y_i' \geq 1/10$. Moreover $y_8' = y_9' = y_{10}' = y_{11}' = (2y_8 + y_{10} + y_{11} + y_{12})/4 =$ $y_9 + (y_{10} + y_{11} + y_{12} - 2y_8)/4 > y_9$ since $7y_1 + 4y_8 < 1$. Thus we assume $7y_1 + 4y_8 \geq 1$. If $7y_1 + 4y_{10} < 1$, then define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,7 \\ (1-7y_1)/4 & \text{for } i = 8,9,10 \\ y_1/2 & \text{for } i = 11 \\ (1-9y_1)/4 & \text{for } i = 12. \end{cases}$$

Then y' is easily shown to be in K_4. Furthermore, y' dom x via $\{5,6,7,8,9,11\}_{y'} | \{7,8,9,10,11,12\}_x$. In fact, $\{5,6,7,8,9,11\}_{y'}$ is

effective since $3y_1' + 2y_8' + y_{11}' = 1/2$. Clearly $y_5' = y_6' = y_7' = y_7 > x_7 \geq x_8 \geq x_9$. $y_8' = y_9' = (1-7y_1)/4 > y_{10} > x_{10} \geq x_{11}$ since $7y_1 + 4y_{10} < 1$. Finally $y_{11}' = y_1/2 > y_{12} > x_{12}$ is shown as follows. Assume $y_{12} \geq y_1/2$. Then we get the contradiction

$$\sum_{i=1}^{12} y_i = 7y_1 + 2y_8 + y_{10} + y_{11} + y_{12} \geq 7y_1 + 2y_8 + 3y_{12} > 1$$

since $7y_1 + 4y_8 \geq 1$. Thus we assume $7y_1 + 4y_{10} \geq 1$. If $7y_1 + 4y_{11} \geq 1$, then $y \in K_3$ and y dom x via $\{6,7,8,9,10,11\}_y | \{7,8,9,10,11,12\}_x$ since $y_1 \geq 1/10$. If $7y_1 + 4y_{11} < 1$, then define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,7 \\ y_{10} - \delta/4 & \text{for } i = 8,9,10 \\ y_i + 3\delta/8 & \text{for } i = 11,12 \end{cases}$$

where $\delta = 7y_1 + 4y_{10} - 1 \geq 0$. If $4y_1' + y_{10}' + y_{12}' \geq 1/2$, then $y' \in K_4$ and y' dom x via $\{4,5,6,7,11,12\}_{y'} | \{7,8,9,10,11,12\}_x$. The effectiveness of $\{4,5,6,7,11,12\}_{y'}$ is shown as follows. If we assume $4y_1' + y_{11}' + y_{12}' \geq 1/2$, then we get the contradiction $5 = 3 + 2 < 21y_1' + 12y_8' + 16y_1' + 4y_{11}' + 4y_{12}' = 4(7y_1' + 3y_8' + y_{11}' + y_{12}') + 9y_1' \leq 5$ since $7y_1' + 4y_8' = 1$ and $y_1' \leq 1/9$. Obviously $y_4' = y_5' = y_6' = y_7' > x_7 \geq x_8 \geq x_9 \geq x_{10}$, $y_{11}' > x_{11}$ and $y_{12}' > x_{12}$. If $4y_1' + y_{10}' + y_{12}' < 1/2$, then define y'' by

$$y_i'' = \begin{cases} y_i' & \text{for } i = 1,\ldots,10 \\ y_i'/2 & \text{for } i = 11 \\ (1-9y_1')/4 & \text{for } i = 12. \end{cases}$$

Then $y'' \in K_4$ and y'' dom x via $\{4,5,6,7,10,12\}_{y''}|\{7,8,9,10,11,12\}_x$. In fact, $\{4,5,6,7,10,12\}_{y''}$ is effective since $4y_1'' + y_{10}'' + y_{12}'' = (7y_1' + 4y_8' + 1)/4 = 1/2$. Clearly $y_4'' = y_5'' = y_6'' = y_7'' > x_7 \geq x_8 \geq x_9 \geq x_{10}$ and $y_{10}'' \geq y_{11} > x_{11}$. Finally $y_{12}'' > y_{12} > x_{12}$ follows from $4y_1'' + y_{10}'' + y_{12}'' = 1/2 > 4y_1'' + y_{10}'' + y_{12}'$.

<u>(iii-II)</u> $\varepsilon = 0$: If $7x_1 + 4x_8 < 1$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/18 & \text{for } i = 1,\ldots,9 \\ x_{10} - \varepsilon'' & \text{for } i = 10 \\ x_{11} - \varepsilon''' & \text{for } i = 11 \\ x_{12} - \varepsilon^{iv} & \text{for } i = 12 \end{cases}$$

where $0 < \varepsilon' < \min(1/9-x_1,\ 1-(7x_1+4x_8))$, $\varepsilon'', \varepsilon''', \varepsilon^{iv} \geq 0$ and $\varepsilon'' + \varepsilon''' + \varepsilon^{iv} = \varepsilon/2$.

If $7x_1 + 4x_8 = 1$ and $7x_1 + 4x_{10} < 1$ then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/22 & \text{for } i = 1,\ldots,11 \\ x_{12} - \varepsilon'/2 & \text{for } i = 12 \end{cases}$$

where $0 < \varepsilon' < \min(1/9-x_1,\ 1-(7x_1+4x_{10}),\ x_{12})$. If $7x_1 + 4x_8 = 1$ and $7x_1 + 4x_{10} = 1$, then $4x_1 + x_8 + x_{12} < 1/2$, $x_{11} > x_{12} > 0$ and $7x_1 + 4x_{11} < 1$. Define y by .

$$\begin{cases} x_i + \varepsilon' & \text{for } i = 1,\ldots,7 \end{cases}$$

$$y_i = \begin{cases} x_8 - 7\varepsilon'/4 & \text{for } i = 8,9,10 \\ x_{11} - 11\varepsilon'/4 & \text{for } i = 11 \\ x_{12} + \varepsilon' & \text{for } i = 12 \end{cases}$$

where $0 < 4\varepsilon' < \min(1/9 - x_1,\ x_{10} - x_{11},\ x_{11} - x_{12},\ 1/2 - (4x_1 + x_8 + x_{12}))$.

If $7x_1 + 4x_8 > 1$ and $7x_1 + 4x_{10} < 1$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/10 & \text{for } i = 1,\dots,7 \\ x_i - \varepsilon'/2 & \text{for } i = 8,9 \\ x_i + \varepsilon'/10 & \text{for } i = 10,11,12 \end{cases}$$

where $0 < 2\varepsilon' < \min(1/9 - x_1,\ 7x_1 + 4x_8 - 1,\ 1 - (7x_1 + 4x_{10}),\ x_8 - x_{10})$.

If $7x_1 + 4x_8 > 1$ and $7x_1 + 4x_{10} = 1$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/10 & \text{for } i = 1,\dots,7 \\ x_8 - \varepsilon'/2 & \text{for } i = 8,9 \\ x_i + \varepsilon'/10 & \text{for } i = 10,11,12 \end{cases}$$

where $0 < 2\varepsilon' < \min(1/9 - x_1,\ x_8 - x_{10},\ 1 - (7x_1 + 4x_{11}))$.

If $7x_1 + 4x_8 > 1$ and $7x_1 + 4x_{10} > 1$, then define

$$y_i = \begin{cases} x_i + \varepsilon'/11 & \text{for } i \neq 10 \\ x_{10} - \varepsilon' & \text{for } i = 10 \end{cases}$$

where $0 < 4\varepsilon' < \min(1/9-x_1,\ x_{10}-x_{11},\ 7x_1+4x_{10}-1)$.

Using these y' we can prove $x \in \operatorname{Dom} \bigcup_{i=1}^{8} K_i$ similarly to that in (iii-I).

<u>Case (iv)</u> $x_7 < 1/10$: Let $\varepsilon = \sum_{i=1}^{9} x_i - (7x_1+2x_9)$. <u>(iv-I)</u> $\varepsilon > 0$: Define y by

$$y_i = \begin{cases} x_7 + \varepsilon' & \text{for } i = 1,\dots,7 \\ x_9 + \varepsilon'' & \text{for } i = 8,9 \\ x_{10} + \varepsilon''' & \text{for } i = 10 \\ x_{11} + \varepsilon^{iv} & \text{for } i = 11 \\ x_{12} + \varepsilon^{v} & \text{for } i = 12 \end{cases}$$

where $\varepsilon', \varepsilon'', \varepsilon''', \varepsilon^{iv}, \varepsilon^{v} \geq 0$, $7\varepsilon' + 2\varepsilon'' + \varepsilon''' + \varepsilon^{iv} + \varepsilon^{v} = \varepsilon$ and $y_1 = \dots = y_7 \leq 1/10$.

If $2y_1 + 4y_8 < 1/2$, then define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\dots,7 \\ (1-4y_1)/8 & \text{for } i = 8,9,10,11 \\ (1-10y_1)/2 & \text{for } i = 12. \end{cases}$$

Then $y' \in K_5$ and y' dom x via $\{6,7,8,9,10,11\}_{y'} | \{7,8,9,10,11,12\}_x$. The effectiveness of $\{6,7,8,9,10,11\}_{y'}$ is trivial since $2y'_1 + 4y'_8 = 1/2$. If $2y_1 + 4y_8 \geq 1/2$ and $2y_1 + 4y_{10} < 1/2$, then define y' by

$$\begin{cases} y_i & \text{for } i = 1,\dots,7 \end{cases}$$

$$y_i' = \begin{cases} (1-4y_1)/8 & \text{for } i = 8,9,10 \\ (2-16y_1)/8 & \text{for } i = 11 \\ (3-28y_1)/8 & \text{for } i = 12. \end{cases}$$

Then $y' \in K_6 \cup K_7$ and y' dom x via $\{5,6,7,8,9,11\}_{y'}|\{7,8,9,10,11,12\}_x$. The effectiveness of $\{5,6,7,8,9,11\}_{y'}$ is obvious. $y_5' = y_6' = y_7' > x_7 \geq x_8 \geq x_9$ and $y_8' = y_9' > x_{10} \geq x_{11}$ are easily shown. Finally, if we assume $y_{11}' < y_{12}$, then we get the contradiction

$$\sum_{i=1}^{12} y_i = 7y_1 + 2y_8 + y_{10} + y_{11} + y_{12} \geq 7y_1 + 2y_8 + 3y_{12}$$

$$= 6y_1 + 3y_{12} + y_1 + 2y_8 = 3(16y_1 + 8y_{12})/8 + (2y_1 + 4y_8)/2 = 1.$$

Thus we must have $y_{11}' \geq y_{12}' > x_{12}$. Now assume $2y_1 + 4y_{10} \geq 1/2$ and $2y_1 + 4y_{11} < 1/2$. If $y_1 \geq 7/72$, then define y' by

$$y_i' = \begin{cases} y_i & \text{for } i = 1,\ldots,7 \\ (1-4y_1)/8 & \text{for } i = 8,9,10 \\ y_{11} + \delta' & \text{for } i = 11 \\ y_{12} + \delta'' & \text{for } i = 12 \end{cases}$$

where $\delta', \delta'' \geq 0$ and $\delta' + \delta'' = y_8 + y_9 + y_{10} - (3-12y_1)/8$. If $4y_1' + y_8' + y_{12}' \geq 1/2$, then $y' \in K_6$ and y' dom x via $\{4,5,6,7,11,12\}_{y'}|\{7,8,9,10,11,12\}_x$. The effectiveness of $\{4,5,6,7,11,12\}_{y'}$ is shown as follows. Assume $4y_1' + y_{11}' + y_{12}' > 1/2$, then

$$\sum_{i=1}^{12} y_i' = 7y_1' + 3y_8' + y_{11}' + y_{12}' = 4y_1' + y_{11}' + y_{12}' + 3y_1' + 3y_8' > 1$$

since $3y_1' + 3y_8' \geq 2y_1' + 4y_8' = 1/2$. If $4y_1' + y_8' + y_{12}' < 1/2$, then define y'' by

$$y_i'' = \begin{cases} y_i' & \text{for } i = 1,\dots,7 \\ (1-4y_1')/8 & \text{for } i = 8,9,10 \\ (2-16y_1')/8 & \text{for } i = 11 \\ (3-28y_1')/8 & \text{for } i = 12. \end{cases}$$

Then $y'' \in K_6$ and y'' dom x via $\{4,5,6,7,10,12\}_{y''}|\{7,8,9,10,11,12\}_x$. Here $y_{12}'' > y_{12}$ follows from the fact that $4y_1' + y_8' + y_{12}' < 1/2 = 4y_1'' + y_8'' + y_{12}''$. If $y_1 < 7/72$, then define the following two imputations y'^{I} and y'^{II}:

$$y'^{I}_i = \begin{cases} y_i & \text{for } i = 1,\dots,7 \\ (1-4y_1)/8 & \text{for } i = 8,9,10 \\ (2-16y_1)/8 & \text{for } i = 11 \\ (3-28y_1)/8 & \text{for } i = 12. \end{cases}$$

$$y'^{II}_i = \begin{cases} y_i & \text{for } i = 1,\dots,7 \\ (1-4y_1)/8 & \text{for } i = 8,9 \\ (2-16y_1)/8 & \text{for } i = 10,11,12. \end{cases}$$

Then $y'^{I} \in K_7$ and $y'^{II} \in K_8$ and y'^{I} dom x via $\{4,5,6,7,10,12\}_{y'^{I}}|\{7,8,9,10,11,12\}_x$ or y'^{II} dom x via $\{4,5,6,7,11,12\}_{y'^{II}}|\{7,8,9,10,11,12\}_x$. In fact, if neither of these holds, then we must have $28y_1 + 8y_{12} \geq 3$ and $16y_1 + 8y_{11} \geq 2$. Hence we get the contradiction

$$\sum_{i=1}^{12} y_i = 7y_1 + 2y_8 + y_{10} + y_{11} + y_{12} = (28y_1 + 8y_8 + 4y_{10} + 4y_{11} + 4y_{12})/4$$

$$= (2y_1 + 4y_8 + 2y_1 + 4y_8 + 2y_1 + 4y_{10} + 8y_1 + 4y_{11} + 14y_1 + 4y_{12})/4 \geq 1$$

where equality holds only if $y \in K_7$ and in this case, x is dominated by y itself via $\{4,5,6,7,10,12\}_y|\{7,8,9,10,11,12\}_x$. If $2y_1 + 4y_{11} \geq 1/2$, then define y' by

$$y'_i = \begin{cases} y_i & \text{for } i = 1,\dots,7 \\ (1-4y_1)/8 & \text{for } i = 8,9,10,11 \\ (1-10y_1)/2 & \text{for } i = 12. \end{cases}$$

Then $y' \in K_5$ and y' dom x via $\{3,4,5,6,7,12\}_{y'}|\{7,8,9,10,11,12\}_x$. Here $y'_{12} > y_{12}$ is shown as follows. Assume $y'_{12} \leq y_{12}$, then we obtain

$$\sum_{i=1}^{12} y_i = 7y_1 + 2y_8 + y_{10} + y_{11} + y_{12} \geq 5y_1 + y_{12} + 2y_1 + 4y_{11} \geq 1$$

where equality holds only if $y_8 = y_9 = y_{10} = y_{11}$ and $2y_1 + 4y_{11} = 1/2$. In this case $y \in K_5$ and y dom x.

<u>(iv-II)</u> $\varepsilon = 0$: If $2x_1 + 4x_8 < 1/2$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/11 & \text{for } i = 1,\dots,11 \\ x_{12} - \varepsilon' & \text{for } i = 12 \end{cases}$$

where ε' is sufficiently small so that $y_1 \leq 1/10$, $y_{12} \geq 0$ and $2y_1 + 4y_8 < 1/2$.

To show $x \in \mathrm{Dom}\, K_5$, use this y and proceed as in (iv-I). Assume $2x_1 + 4x_8 = 1/2$ and $2x_1 + 4x_{10} < 1/2$. If $x_{10} < 1/18$, then $y = (\underbrace{1/9,\ldots,1/9}_{7}, 1/18, 1/18, 1/18, 1/18, 0)$ dom x via

$\{5,6,7,8,9,10\}_y \,|\, \{7,8,9,10,11,12\}_x$ and $y \in K_2$. Thus we assume $x_{10} \geq 1/18$. Since $x \notin \bigcup_{i=1}^{8} K_i$, we must have $x_{10} > x_{12}$. Now define y by

$$y_i = \begin{cases} x_i + \varepsilon'/11 & \text{for } i = 1,\ldots,11 \\ x_{12} - \varepsilon' & \text{for } i = 12 \end{cases}$$

where ε' is sufficiently small so that $y_1 \leq 1/10$, $y_{12} \geq 0$, $2y_1 + 4y_{10} < 1/2$ and $6y_1 + 3x_{12} < 3/4$. Then by using this y, $y \in \mathrm{Dom}(K_6 \cup K_7)$ is shown as in (iv-I). Assume $2x_1 + 4x_8 = 2x_1 + 4x_{10} = 1/2$. Here we note that $2x_1 + 4x_{11} < 1/2$ since $x \notin \bigcup_{i=1}^{8} K_i$. First, assume $x_1 \geq 7/72$. Then we must have $4x_1 + x_8 + x_{12} < 1/2$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon' & \text{for } i = 1,\ldots,7 \\ x_i - \varepsilon'/2 & \text{for } i = 8,9,10 \\ x_{11} - 13\varepsilon'/2 & \text{for } i = 11 \\ x_{12} + \varepsilon' & \text{for } i = 12 \end{cases}$$

where ε' is sufficiently small so that $y_1 \leq 1/10$, $y_{11} \geq y_{12}$, $2y_1 + 4y_{11} < 1/2$ and $4y_1 + y_8 + y_{12} < 1/2$. Then $x \in \mathrm{Dom}\, K_6$ is deduced as in (iv-II). Next, we assume $x_1 < 7/72$. Then we must have $4x_1 + x_8 + x_{12} > 1/2$. Define y by

$$y_i = \begin{cases} x_i + 3\varepsilon'/7 & \text{for } i = 1,\ldots,7 \\ x_i & \text{for } i = 8,9 \end{cases}$$

$$\quad (x_{10}+x_{11}+x_{12})/3-\varepsilon' \quad \text{for } i = 10,11,12$$

where ε' is sufficiently small so that $y_1 \leq 7/72$ and $(x_{10}+x_{11}+x_{12})/3 > x_{11}$. Then $y \in K_8$ and y dom x via $\{4,5,6,7,11,12\}_y | \{7,8,9,10,11,12\}_x$. Finally we suppose $2x_1 + 4x_8 > 1/2$. If $x_8 > x_{10}$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/10 & \text{for } i \neq 8,9 \\ x_i - \varepsilon'/2 & \text{for } i = 8,9 \end{cases}$$

where ε' is sufficiently small so that $y_1 \leq 1/10$, $y_8 > y_{10}$ and $2y_1 + 4y_8 > 1/2$. If $x_8 = x_{10} > x_{11}$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/9 & \text{for } i \neq 8,9,10 \\ x_i - \varepsilon'/3 & \text{for } i = 8,9,10 \end{cases}$$

where ε' is sufficiently small so that $y_1 < 1/10$, $y_8 > y_{11}$ and $2y_1 + 4y_8 > 1/2$. If $x_8 = x_{10} = x_{11}$, then define y by

$$y_i = \begin{cases} x_i + \varepsilon'/8 & \text{for } i \neq 8,9,10,11 \\ x_i - \varepsilon'/4 & \text{for } i = 8,9,10,11 \end{cases}$$

where ε' is sufficiently small so that $y_1 \leq 1/10$, $y_8 > y_{12}$, and $2y_1 + 4y_8 > 1/2$. By using such y we obtain $x \in \text{Dom} \bigcup_{i=1} K_i$ in a manner similar to (iv-I). □

As a result of Theorems 6.6 and 6.7, symmetric stable sets for all $(n;k)_h$ games with $k \leq 6$ have been obtained. The author also

found symmetric stable sets for $(14;7)_h$, $(16;8)_h$ and $(18;9)_h$ games which are extensions of these two theorems but these are not described in this work. So games marked by "$\bar{\ }$" in Figure 6.1 have also been solved.

6.3. The Uniqueness of K_h

Hart's second open question will be answered by the following.

Theorem 6.8: K_h defined in Theorem 6.1 is the unique symmetric stable set for $(n;k)_h$ games with $q \geq 2$ if and only if (a). $r = 0$ and $n \geq (q+1)(k-1) + 1$ or (b). $r \geq 1$ and $n \geq (q+1)(k-1)$.

Proof: Sufficiency: We first note that we can assume $k \geq 4$ since if $k \leq 3$, then the condition in Theorem 6.2 is always satisfied. The proof will proceed from the following sequence of claims.

Claim 1: Define $a = (k-2)(n-k-1)/(k-1)(n-2k+2)$ and $b = (q+1-k)/q(k-1)(n-2k+2)$. Then (a). $a > 0$, (b). $a < 1$ if $q+1 > k$ and (c). $b > 0$ if and only if $q+1 > k$.

Proof of Claim 1: (a) and (c) are obvious. (b) is shown by a straightforward calculation, i.e.,

$$a-1 = (k-2)(n-k+1)/(k-1)(n-2k+2) - 1$$

$$\leq (k^2-k-(q+1)(k-1))/(k-1)(n-2k+2)$$

$$= (k-(q+1))/(n-2k+2). \qquad \square$$

Remark: If $n = (q+1)(k-1)$ then (b) becomes "$a < 1$ if and only if $q+1 > k$."

Case (i) $n \geq (q+1)(k-1)+1$: Let $u = (q-1)/q(n-2k+2)$ and $c_m = a^m u + (a^{m-1} + \ldots + 1)b$ for $m = 0,1,2,\ldots$. For convenience, let $c_0 = u$. Then we come to the following claims.

Claim 2: $1/qk \leq u < 1/(n-k+1)$.

Proof of Claim 2: $1/qk - u = 1/qk-(q-1)/q(n-2k+2) \leq (2-q)/qk(n-2k+2) \leq 0$. $u - 1/(n-k+1) \leq -1/q(n-2k+2)(n-k+1) < 0$. □

Claim 3: c_m is monotone decreasing.

Proof of Claim 3: $c_m - c_{m-1} = a^{m-1}\{(a-1)u+b\} \leq a^{m-1}(-k+2)/q(k-1)(n-2k+2)^2 < 0$. □

Claim 4:

$$\lim_{m\to\infty} c_m \begin{cases} = 1/qk & \text{if } r = 0 \\ < 1/qk & \text{if } r \geq 1. \end{cases}$$

Proof of Claim 4: First assume $r = 0$. Then $q > k+1$ and thus $0 < a < 1$. Therefore

$$\lim_{m\to\infty} c_m = \lim_{m\to\infty} \{a^m u + (a^{m-1} + \ldots + 1)b\} = 1/qk.$$

Now let $r \geq 1$. If $a < 1$, then

$$\lim_{m\to\infty} c_m = b/(1-a) = (q+1-k)/(qk(q+1-k)+qr) < 1/qk.$$

If $a \geq 1$, then

$$\lim_{m\to\infty} c_m = u + ((a-1)u+b) \lim_{m\to\infty} (a^{m-1} + \ldots + 1) = -\infty$$

since $(a-1)u + b < 0$, as shown in the proof of Claim 3. $\square$

Now define

$$A_0 = \langle \{x \in \lceil A \rceil \mid x_1 \geq 1/qk\} \rangle,$$

$$A_{0,m} = \langle \{x \in \lceil A_0 \rceil \mid x_1 \geq c_m\} \rangle \quad \text{for } m = 0,1,2,\ldots$$

and

$$A_1 = A - A_0.$$

We are now ready to state and prove the next claim which plays a crucial part of the proof.

<u>Claim 5</u>: Let K be any symmetric stable set. For any $m(= 0,1,2,\ldots)$, if $x \in A_{0,m} \cap K$ then $x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_n$.

<u>Proof of Claim 5</u>: This proof will proceed by induction on m. Assume $m = 0$ and take any $x \in A_{0,0} \cap K$. Suppose $x_{n-k+2} > x_n$ and define y by

$$y_i = \begin{cases} x_{n-k+1} + \varepsilon & \text{for } i = 1,\ldots,n-k+1 \\ x_n + \varepsilon & \text{for } i = n-k+2,\ldots,n \end{cases}$$

where $n\varepsilon = \sum_{i=n-k+2}^{n} x_i - (k-1)x_n$. Then y dom x via $\{1,\ldots,k-1,n\}_y \mid \{n-k+1,\ldots,n\}_x$. The effectiveness of $\{1,\ldots,k-1,n\}_y$ is shown as follows. Assume $(k-1)/y_1 + y_n > 1/q$. If $q+1 > k$, then

$$\sum_{i=1}^{n} y_i = (n-k+1)y_1 + (k-1)y_n = (k-1)((k-1)y_1 + y_n) + ((n-k+1)-(k-1)^2)y_1$$

$$= (k-1)((k-1)y_1 + y_n) + (k(q+1-k)+r)y_1 > 1.$$

If $q+1 \leq k$, then

$$\sum_{i=1}^{n} y_i = (n-k+1)y_1 + (k-1)y_n = q((k-1)y_1 + y_n) + (n-k+1-q(k-1))y_1$$

$$+ (k-q-1)y_n > 1$$

since $n-k+1-q(k-1) \geq 1$. Clearly $y_1 = \dots = y_{k-1} > x_{n-k+1} \geq \dots \geq x_{n-1}$ and $y_n > x_n$. Since $x \in K$, there must exist some $z \in K$ such that z dom y via $S_z|\{n-k+1,\dots,n\}_y$. This z satisfies $z_1 = \dots = z_{n-k+1} > y_{n-k+1} > x_{n-k+1} \geq \dots \geq x_{n-1}$ and $z_{n-k+2} > y_n > x_n$. Hence if $\{1,\dots,k-1,n-k+2\}_z$ is effective, then we get z dom x via $\{1,\dots,k-1, n-k+2\}_z|\{n-k+1,\dots,n\}_x$ which contradicts the fact that $z,x \in K$. Suppose $(k-1)z_1 + z_{n-k+2} > 1/q$. Then

$$\sum_{i=1}^{n} z_i \geq (n-k+1)z_1 + z_{n-k+2} = (k-1)z_1 + z_{n-k+2} + (n-2k+2)z_1$$

$$> 1/q + (n-2k+2)\cdot(q-1)/q(n-2k+2) = 1.$$

Therefore we have shown that the claim is true for $m = 0$.

Suppose the claim to be true for $m = k$ and take any $x \in ((A_{0,k+1} - A_{0,k}) \cap K)$. Here if $A_{0,k+1} - A_{0,k} = \emptyset$, then no proof is required. So we assume $A_{0,k+1} - A_{0,k} \neq \emptyset$. Assume $x_{n-k+2} > x_n$

and define y by

$$y_i = \begin{cases} x_{n-k+1} + \varepsilon & \text{for } i = 1,\dots,n-k+1 \\ x_n + \varepsilon & \text{for } i = n-k+2,\dots,n \end{cases}$$

where $n\varepsilon = \sum_{i=n-k+2}^{n} x_i - (k-1)x_n$. Then y dom x via $\{1,\dots,k-1,n\}_y|\{n-k+1,\dots,n\}_x$ as shown above. Thus there must exist some $z \in K$ such that z dom y. If $z \in A_{0,k} \cap K$, then by the induction hypothesis we must have $z_{n-k+2} = \dots = z_n$ and thus z d$\not{o}$m y. Assume $z \in (A_{0,k+1} - A_{0,k}) \cap K$. Then $z_1 < c_k$. Now define z' by

$$z'_i = \begin{cases} c_k & \text{for } i = 1,\dots,n-k+1 \\ (1-(n-k+1)c_k)/(k-1) & \text{for } i = n-k+2,\dots,n. \end{cases}$$

Then $z' \in K$ since there is no imputation in K which dominates z' by the induction hypothesis. If $z_n < z'_n$, then z' dom z via $\{1,\dots,k-1,n\}_{z'}|\{n-k+1,\dots,n\}_z$ which is contrary to the fact that $z', z \in K$. Assume $z_n \geq z'_n$. Then we have z dom x via $\{1,\dots,k-1,n-k+2\}_z|\{n-k+1,\dots,n\}_x$. The effectiveness of $\{1,\dots,k-1,n-k+2\}_z$ is proved in the following way. Assume $(k-1)z_1 + z_{n-k+2} > 1/q$. Then

$$\sum_{i=1}^{n} z_i \geq (n-k+1)z_1 + z_{n-k+2} + (k-2)z_n$$

$$= (k-1)z_1 + z_{n-k+2} + (n-2k+2)z_1 + (k-2)z_n$$

$$> 1/q + (n-2k+2)(az'_1 + b) + (k-2)(1-(n-k+1)z'_1)/(k-2)$$

$$= 1/q + z'_1(a(n-2k+2) - (k-2)(n-k+1)/(k-1) + b(n-2k+2)$$

$$+ (k-2)/(k-1) = 1/q + (q+1-k)/q(k-1) + (k-2)/(k-1) = 1.$$

Evidently $z_1 = \dots = z_{k-1} > x_{n-k+1} \geq \dots \geq x_{n-1}$ and $z_{n-k+2} > x_n$. Thus we obtain the contradiction, since $x, z \in K$. Therefore we obtain $x_{n-k+2} = \dots = x_n$. □

For $r = 0$, this claim, together with Claims 3 and 4, shows the uniqueness of K_h. For $r \geq 1$, we need to prove one more claim in order to establish the uniqueness.

Claim 6: For any symmetric stable set K, $A_1 \cap K = \emptyset$.

Proof of Claim 6: Pick any $x \in A_1$ and define y by

$$y_i = \begin{cases} 1/qk & \text{for } i = 1,\dots,n-k+1 \\ (k-(r+1))/qk(k-1) & \text{for } i = n-k+2,\dots,n. \end{cases}$$

Then $y \in K$ and y dom x via $\{1,\dots,k\}_y|\{n-k+1,\dots,n\}_x$. □

Case (ii) $r \geq 1$ and $n = (q+1)(k-1)$: Let $u' = (q-1)/q(n-2k+3)$ and $c'_m = a^m u' + (a^{m-1} + \dots + 1)b$ for $m = 0,1,2,\dots$. For convenience, let $c'_0 = u'$. Then we have the following claims which are analogous to Claims 2,3 and 4 in Case (i).

Claim 2': $1/qk \leq u' < 1/(n-k+1)$.

Proof of Claim 2': $1/qk - u' = (2-q)/qk(n-2k+3) \leq 0$. $u' - 1/(n-k+1) = -1/(n-k+1)(n-2k+3) < 0$. □

Claim 3': c'_m is monotone decreasing.

Proof of Claim 3': $c'_m - c'_{m-1} = a^{m-1}((a-1)u'+b) =$
$a^{m-1}((k-(q+1))(q-1)/q(n-2k+2)(n-2k+3) + (q+1-k)/q(k-1)(n-2k+2)) = a^{m-1}\cdot$
$(q+1-k)/q(k-1)(n-2k+2)(n-2k+3) < 0$, since $n = (q+1)(k-1)$ and $r \geq 1$. □

Claim 4': $\lim_{m\to\infty} c'_m = -\infty$.

Proof of Claim 4': From the remark in the proof of Claim 1, we have $a > 1$. Hence the proof is exactly the same as in Claim 4. □

Now define A_0 and A_1 as in Case (i). Define additional sets $A'_{0,m}$ $(m = 0,1,2,\ldots)$ by

$$A'_{0,m} = \langle\{x \in \lceil A_0 \rceil \mid x_1 \geq c'_m\}\rangle \quad \text{for} \quad m = 0,1,2,\ldots .$$

An analogue of Claim 5 is given by the next claim.

Claim 5': Let K be any symmetric stable set. For any $m = 0,1,2,\ldots$, if $x \in A'_{0,m} \cap K$ then $x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_n$.

Proof of Claim 5': We will proceed by induction. Assume $m = 0$ and take any $x \in A'_{0,0} \cap K$. We will first show that $x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_{n-1} \geq x_n$. Suppose $x_{n-k+2} > x_{n-1}$ and define y by

$$y_i = \begin{cases} x_1 + \varepsilon & \text{for } i = 1,\ldots,n-k+1 \\ x_{n-1} + \varepsilon & \text{for } i = n-k+2,\ldots,n-1 \\ x_n + \varepsilon & \text{for } i = n \end{cases}$$

where $n\varepsilon = \sum_{i=n-k+2}^{n-1} x_i - (k-2)x_{n-1}$. Then y dom x via $\{1,\ldots,k-1,n\}_y \mid \{n-k+1,\ldots,n\}_x$. The effectiveness of $\{1,\ldots,k-1,n\}_y$ is shown as follows. Assume $(k-1)y_1 + y_n > 1/q$. Then we get the contradiction

$$\begin{aligned}\sum_{i=1}^{n} y_i &= (n-k+1)y_1 + (k-2)y_{n-k+2} + y_n \geq (n-k+1)y_1 + (k-1)y_n \\ &= q((k-1)y_1 + y_n) + (n-k+1-q(k-1))y_1 + (k-1-q)y_n > 1\end{aligned}$$

since $n-k+1-q(k-1) = 0$ and $k > q+1$. Since $x \in K$, there must exist some $z \in K$ such that z dom x via $S_z|\{n-k+1,\ldots,n\}_y$. Two cases must be considered. (a). $S_z = \{1,\ldots,k-1,i\}_z$ where $n-k+2 \leq i \leq n$. In this case, we have $z_1 = \ldots = z_{k-1} > y_1 > x_{n-k+1} \geq \ldots \geq x_{n-1}$ and $z_i > y_n > x_n$. Hence z dom x via $S_z|\{n-k+1,\ldots,n\}_x$ which contradicts the fact that $x, z \in K$. (b). $S_z = \{1,\ldots,k-j,i(1),\ldots,i(j)\}_z$ where $2 \leq j \leq k-1$ and $n-k+2 \leq i(1) \leq \ldots \leq i(j) \leq n$. In this case, we have $z_1 = \ldots = z_{k-2} > y_1 > x_{n-k+1} \geq \ldots \geq x_{n-2}$, $z_{n-k+2} \geq z_{i(1)} > y_{n-1} > x_{n-1}$ and $z_{n-k+3} \geq z_{i(2)} > y_n > x_n$. Moreover $\{1,\ldots,k-2,n-k+2,n-k+3\}_z$ is effective. In fact, if we suppose $(k-2)z_1 + z_{n-k+2} + z_{n-k+3} > 1/q$, then we get the contradiction

$$\sum_{i=1}^{n} z_i = (n-k+1)z_1 + \sum_{i=n-k+2}^{n} z_i \geq (n-k+1)z_1 + z_{n-k+2} + z_{n-k+3}$$

$$= (n-2k+3)z_1 + (k-2)z_1 + z_{n-k+2} + z_{n-k+3} > 1$$

since $z_1 > x_1 > u' = (q-1)/q(n-2k+3)$. Therefore z dom x via $\{1,\ldots,k-2,n-k+2,n-k+3\}_z|\{n-k+1,\ldots,n\}_x$ which contradicts the fact that $x, z \in K$. Thus we have shown that $x_1 = \ldots = x_{n-k+1} \geq x_{n-k+2} = \ldots = x_{n-1} \geq x_n$.

Now, we will show that $x_{n-1} = x_n$. Suppose $x_{n-1} > x_n$ and define y by

$$y_i = \begin{cases} x_1 + \varepsilon & \text{for } i = 1,\ldots,n-k+1 \\ x_n + \varepsilon & \text{for } i = n-k+2,\ldots,n \end{cases}$$

where $n\varepsilon = \sum_{i=n-k+2}^{n} x_i - (k-1)x_n$. Then y dom x via

$\{1,\dots,k-1,n\}_y | \{n-k+1,\dots,n\}_x$ since $y_1 = \dots = y_{k-1} > x_{n-k+1} \geq \dots \geq x_{n-1}$ and $y_n > x_n$. The effectiveness of $\{1,\dots,k-1,n\}_y$ was already shown above. Thus there is some $z \in K$ such that z dom y. As shown above this z must satisfy $z_1 = \dots = z_{n-k+1} \geq z_{n-k+2} = \dots = z_{n-1} \geq z_n$. Now we will show that $\{1,\dots,k-1,n-k+2\}_z$ is effective. Suppose $(k-1)z_1 + z_{n-k+2} > 1/q$. Then we obtain

$$\sum_{i=1}^{n} z_i = (n-k+1)z_1 + (k-2)z_{n-k+2} + z_n \geq (n-k+1)z_1 + (k-2)z_{n-k+2}$$

$$= q((k-1)z_1 + z_{n-k+2}) + (n-k+1-q(k-1))z_1 + (k-2-q)z_{n-k+2} > 1$$

since $n = (q+1)(k-1)$ and $k > q+1$. Furthermore we have $z_1 = \dots = z_{k-1} > x_{n-k+1} \geq \dots \geq x_{n-1}$ and $z_{n-k+2} > y_n > x_n$. Hence z dom x via $\{1,\dots,k-1,n-k+2\}_z | \{n-k+1,\dots,n\}_x$ which contradicts the fact $x,z \in K$. Thus we have shown that the claim is true for $m = 0$.

The rest of the proof proceeds along the same lines as in Claim 5. □

The uniqueness of K_h in Case (ii) is obtained from Claims 3', 4', 5' and 6.

<u>Necessity</u>: This is clear from Theorem 6.1 and Remark (b) in Theorem 6.3. □

6.4 <u>Semi-symmetric Stable Sets</u>

We conclude this chapter by stating the following theorem which shows that $(n;k)_h$ games always have semi-symmetric stable sets as defined in Chapter IV. Before stating the theorem, we note that in this section for $x \in A$, the coordinates of x are not necessarily arranged into nonincreasing order.

Theorem 6.9: Let $\{S_1,\ldots,S_{q+1}\}$ be a partition of N satisfying $|S_j| = k$ for $j = 1,\ldots,q$ and $|S_{q+1}| = r$. Define

$$K_j = \{x \in A \mid \sum_{i\in S_j} x_i = 1/q;\ x_i = 1/qk \text{ for all } i \in N - S_j - S_{q+1};$$
$$x_i = 0 \text{ for all } i \in S_{q+1}\} \text{ for } j = 1,\ldots,q.$$

Then $\bigcup_{j=1}^{q} K_j$ is a stable set for $(n;k)_h$ games with $n = qk + r$.

Proof: We will omit this proof since it is similar to that of Theorem 4.4. □

Remark: When $q = 1$, the above $\bigcup_{j=1}^{q} K_j$ turns out to be

$$\{x \in A \mid \sum_{i\in S_1} x_i = 1;\ x_i = 0 \text{ for all } i \in N - S_1\}.$$

This is a well known "main simple stable set".

CHAPTER VII

SYMMETRIC SUBSOLUTIONS

In [30], A. Roth introduced an interesting generalization of the stable set, called a subsolution, and proved its existence for all games with nonempty core. The aim of this chapter is to determine the minimal nonempty symmetric subsolutions for symmetric games and to investigate how they differ from stable sets. As a result, the coincidence of cores with supercores will be shown when games are symmetric.

7.1 Preliminaries

We begin this chapter with a brief review of Roth's results.

Definition 7.1: A subset K_{sub} of A is said to be a subsolution if it satisfies the following two conditions:

$$\text{(a). } K_{sub} \subseteq U(K_{sub}) \text{ and } \text{(b). } K_{sub} = U^2(K_{sub}) = U(U(K_{sub})).$$

Recall that for $B \subseteq A$, $U(B) = A - \text{Dom } B$.

Theorem 7.1 (Roth): If the core is nonempty, then there always exists a nonempty subsolution.

Theorem 7.2 (Roth): If the core is nonempty, then the intersection of all nonempty subsolutions is nonempty and is itself a subsolution, which is called the supercore.

It is not true, in general, that cores coincide with supercores. See Example 5.3 (i.e., the ten person game with no stable set presented by W. Lucas) in [30]. However when games are symmetric, their coincidence will be proved in Section 7.5 of this chapter.

7.2 3-Person and 4-Person Symmetric Games

As we did for stable sets, let us first study subsolutions for 3-person and 4-person symmetric games. The symbol $K_{sub,m,s}$ will be used to denote a minimal nonempty symmetric subsolution.

7.2.1 3-Person Symmetric Games (3;2)

$v(2) = 1$:

$$K_{sub,m,s} = \langle (1/2,\ 1/2,\ 0) \rangle.$$

$2/3 < v(2) < 1$:

$$K_{sub,m,s} = \langle (v(2)/2,\ v(2)/2,\ 1-v(2)) \rangle.$$

$v(2) = 2/3$:

$$K_{sub,m,s} = C = (1/3,\ 1/3,\ 1/3).$$

$v(2) < 2/3$:

$$K_{sub,m,s} = C.$$

These cases are shown in Figure 7.1.

7.2.2 (4;3) Games

$v(3) = 1$:

$$K_{sub} = \langle \{x \in \lceil A \rceil \mid x_1 = x_2 \geq x_3 = x_4\} \rangle.$$

$3/4 < v(3) < 1$:

$$K_{sub} = \langle \{x \in \lceil A \rceil \mid x_1 = x_2 \geq x_3 = x_4 \geq 1-v(3)\} \rangle.$$

$v(3) = 3/4$:

$$K_{sub,m,s} = C = (1/4,\ 1/4,\ 1/4,\ 1/4).$$

$v(3) < 3/4$:

$$K_{sub,m,s} = C.$$

The author only conjectures that the above K_{sub} for $v(3) \geq 3/4$ is a minimal symmetric nonempty subsolution. It has not yet been proved. These are illustrated in Figure 7.2.

7.2.3 (4;2) Games

$2/3 \leq v(2)$:

$$K_{sub,m,s} = \langle (1/3,\ 1/3,\ 1/3,\ 0) \rangle.$$

$1/2 < v(2) < 2/3$:

$$K_{sub,m,s,} = \langle (v(2)/2,\ v(2)/2,\ v(2)/2,\ 1\text{-}3v(2)/2) \rangle.$$

$v(2) = 1/2$:

$$K_{sub,m,s} = C = (1/4,\ 1/4,\ 1/4,\ 1/4).$$

$v(2) < 1/2$:

$$K_{sub,m,s} = C.$$

Figure 7.3 illustrates these sets. As is easily seen from figures, $K_{sub,m,s}$ is obtained from K_{sym} by removing its "bargaining curves." Now let us generalize the results obtained above.

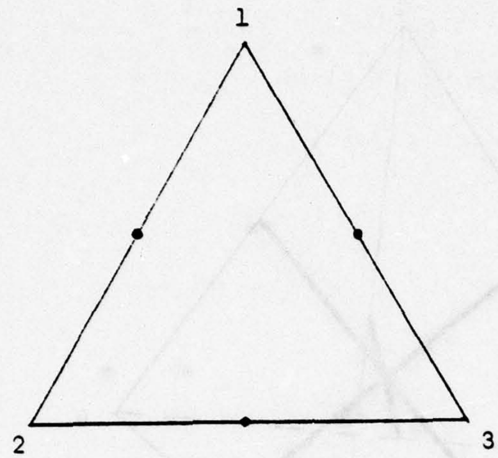
7.3. (n;2) Games

As a generalization of the results obtained for (3;2) and (4;2), we have the next theorem.

Theorem 7.3: Consider (n;2) games with empty core. Define

$$K_{sub} = \langle (v(2)/2, \ldots, v(2)/2,\ 1-(n-1)v(2)/2) \rangle.$$

$v(2) = 1$



$\frac{2}{3} < v(2) < 1$

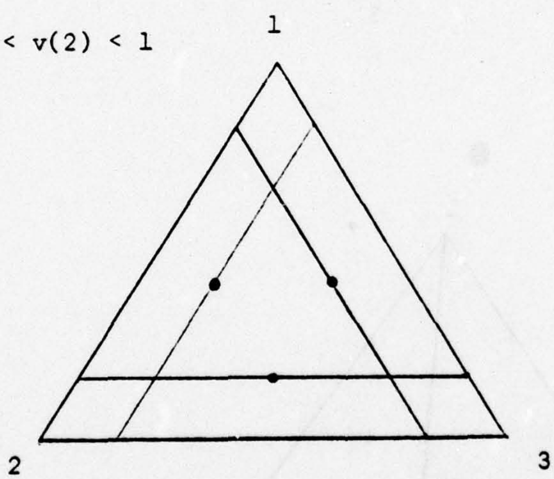
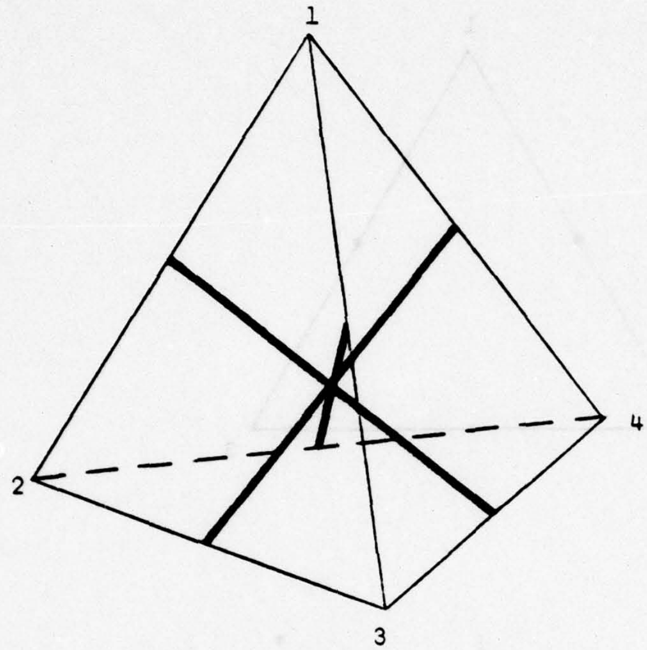


Figure 7.1 Symmetric subsolutions for (3;2)

$v(3) = 1$



$\frac{3}{4} < v(3) < 1$

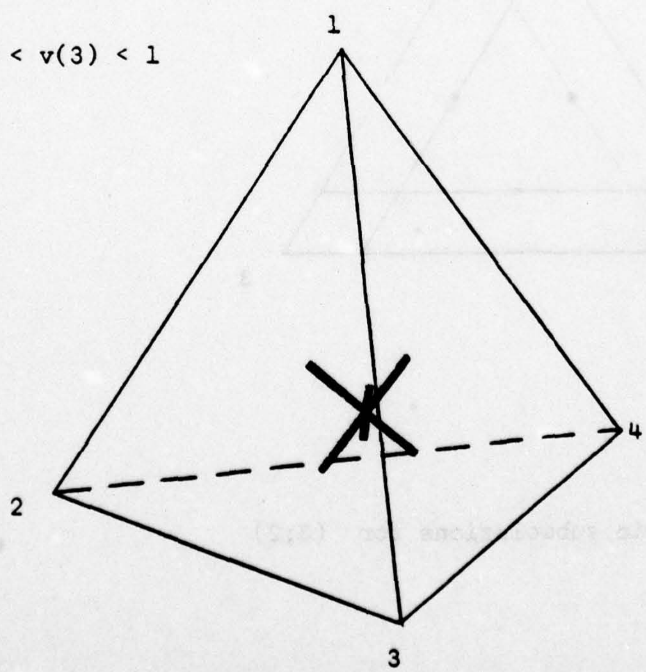
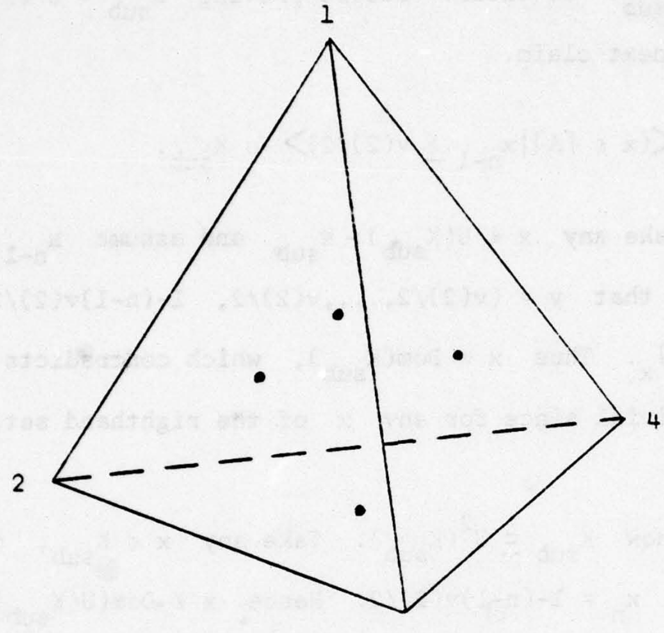


Figure 7.2 Symmetric subsolutions for (4;3)

2/3 $\leq$ v(2)



1/2 < v(2) < 2/3

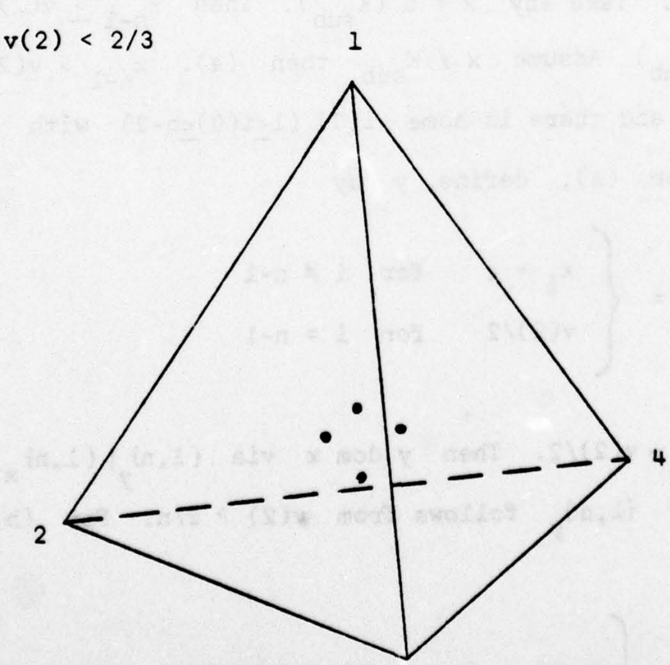


Figure 7.3 Symmetric subsolutions for (4;2)

Then K_{sub} is a minimal nonempty symmetric subsolution.

<u>Proof</u>: $K_{sub} \subseteq U(K_{sub})$ is clear. Before proving $K_{sub} = U^2(K_{sub})$ we will consider the next claim.

<u>Claim</u>: $U(K_{sub}) = \langle\{x \in \lceil A\rceil \mid x_{n-1} \geq v(2)/2\}\rangle \cup K_{sub}$.

<u>Proof of Claim</u>: Take any $x \in U(K_{sub}) - K_{sub}$ and assume $x_{n-1} < v(2)/2$. Then it is obvious that $y = (v(2)/2,\ldots,v(2)/2,\ 1-(n-1)v(2)/2)$ dom x via $\{1,2\}_y | \{n-1,n\}_x$. Thus $x \in \mathrm{Dom}(K_{sub})$, which contradicts $x \in U(K_{sub})$. The converse is trivial since for any x of the righthand set, $x_{n-1} \geq v(2)/2$. □

Now we will show $K_{sub} \subseteq U^2(K_{sub})$. Take any $x \in K_{sub}$, then $x_{n-1} = v(2)/2$ and $x_n = 1-(n-1)v(2)/2$. Hence $x \notin \mathrm{Dom}(U(K_{sub}))$ since for any $y \notin U(K_{sub})$, $y_{n-1} \geq v(2)/2$ and thus $y_n \leq 1-(n-1)v(2)/2$ from the above claim. Therefore $x \in U^2(K_{sub})$. Next, $K_{sub} \supseteq U^2(K_{sub})$ is shown as follows. Take any $x \in U^2(K_{sub})$. Then $x_{n-1} \geq v(2)/2$ since $U(K_{sub}) \supseteq K_{sub}$. Assume $x \notin K_{sub}$ then (a). $x_{n-1} > v(2)/2$ or (b). $x_{n-1} = v(2)/2$ and there is some $i(0)$ $(1 \leq i(0) \leq n-2)$ with $x_{i(0)} > x_{i(0)+1}$. For (a), define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i \neq n-1 \\ v(2)/2 & \text{for } i = n-1 \end{cases}$$

where $(n-1)\varepsilon = x_{n-1} - v(2)/2$. Then y dom x via $\{1,n\}_y | \{1,n\}_x$. The effectiveness of $\{1,n\}_y$ follows from $v(2) > 2/n$. For (b), define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i \neq i(0) \\ x_i-(n-1)\varepsilon & \text{for } i = i(0) \end{cases}$$

where $0 < \varepsilon < (x_{i(0)} - x_{i(0)+1})/n$. Then y dom x via $\{1,n\}_y \mid \{1,n\}_x$. In either case we obtain $x \in \text{Dom}\,(U(K_{sub}))$ which contradicts $x \in U^2(K_{sub})$. Finally, minimality is easily seen.

7.4 (n;n-1) Games

Theorem 7.4: Consider (n;n-1) games with empty cores. Define

$$K_{sub} = \begin{cases} \langle\{x \in \ulcorner A \urcorner \mid x_1 = x_2 \geq \dots \geq x_{n-1} = x_n \geq 1-v(n-1)\}\rangle & \text{if } n \text{ is even,} \\ \langle\{x \in \ulcorner A \urcorner \mid x_1 = x_2 \geq \dots \geq x_{n-2} = x_{n-1} \geq x_n = 1-v(n-1)\}\rangle & \text{if } n \text{ is odd.} \end{cases}$$

Then K_{sub} is a symmetric subsolution.

Proof: Let $\omega = 1-v(n-1)$ and define

$$A_j = \langle\{x \in \ulcorner A \urcorner \mid x_1 \geq \dots \geq x_{n-j} \geq \omega > x_{n-j+1} \geq \dots \geq x_n\}\rangle \text{ for } j = 0,1,\dots,n-1.$$

Then it is easily shown that $\{A_0, A_1, \dots, A_{n-1}\}$ is a partition of A and $K_{sub} \subseteq A_0$. We will first prove the following claims which will be useful in showing that K_{sub} is a subsolution.

Claim 1: $K_{sub} \cap \text{Dom}(K_{sub}) = \emptyset$ and thus $K_{sub} \subseteq U(K_{sub})$.

Proof of Claim 1: This follows from Theorem 3.2. □

Claim 2: $A_0 = K_{sub} \cup \text{Dom}(K_{sub})$.

Proof of Claim 2: We will show $A_0 - K_{sub} \subseteq \text{Dom}(K_{sub})$. Take any $x \in A_0 - K_{sub}$.

Case (i) n is even: There is some odd $i(0)$ such that $x_{i(0)} > x_{i(0)+1}$. Define y by

$$y_i = y_{i+1} = x_{i+1} + \varepsilon \quad \text{for } i = 1,3,\ldots,n-1$$

where $n\varepsilon = \sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n/2} x_{2i}$. Then $y \in K_{sub}$ and y dom x via $\{1,\ldots,n-1\}_y|\{2,\ldots,n\}_x$. Therefore $x \in \mathrm{Dom}(K_{sub})$.

Case (ii) n is odd: Let $n\varepsilon = (\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{(n-1)/2} x_{2i}) + (x_n - \omega)$ and define y by

$$y_i = y_{i+1} = x_{i+1} + \varepsilon \quad \text{for } i = 1,3,\ldots,n-2, \text{ and}$$

$$y_n = \omega.$$

Then $y \in K_{sub}$ and y dom x via $\{1,\ldots,n-1\}_y|\{2,\ldots,n\}_x$. Therefore $x \in \mathrm{Dom}(K_{sub})$. $\square$

Claim 3: $A_{n-1} \subseteq \mathrm{Dom}(K_{sub})$.

Proof of Claim 3: Take any $x \in A_{n-1}$. Then $x_1 \geq \omega > x_2 \geq \ldots \geq x_n$.

Case (i) n is even: Define y by

$$y_i = \begin{cases} (1-2\omega)/(n-2) & \text{for } i = 1,\ldots,n-2 \\ \omega & \text{for } i = n-1,n. \end{cases}$$

Then $y \in K_{sub}$ and y dom x via $\{1,\ldots,n-1\}_y|\{2,\ldots,n\}_x$.

Case (ii) n is odd: Define y by

$$y_i = \begin{cases} (1-\omega)/(n-1) & \text{for } i = 1,\ldots,n-1 \\ \omega & \text{for } i = n. \end{cases}$$

Then $y \in K_{sub}$ and y dom x via $\{1,\ldots,n-1\}_y|\{2,\ldots,n\}_x$. In either case, we obtain $x \in \mathrm{Dom}(K_{sub})$. □

Claim 4: Take any A_j ($j = 1,\ldots,n-2$) and any x of A_j.

(a). n is even: $x \in \mathrm{Dom}(K_{sub})$ if and only if

$$\sum_{i=1}^{(n-j)/2} x_{2i} < (1-j\omega)/2 \qquad \text{if } j \text{ is even, and}$$

$$\sum_{i=1}^{(n-(j+1))/2} x_{2i} < (1-(j+1)\omega)/2 \qquad \text{if } j \text{ is odd.}$$

(b). n is odd: $x \in \mathrm{Dom}(K_{sub})$ if and only if

$$\sum_{i=1}^{(n-(j+1))/2} x_{2i} < (1-(j+1)\omega)/2 \qquad \text{if } j \text{ is even, and}$$

$$\sum_{i=1}^{(n-j)/2} x_{2i} < (1-j\omega)/2 \qquad \text{if } j \text{ is odd.}$$

Proof of Claim 4: We first assume n to be even.

Sufficiency: Case (i) j is even: Define y by

$$y_i = y_{i+1} = x_{i+1} + \varepsilon \quad \text{for } i = 1,3,\ldots,n-j-1, \text{ and}$$

$$y_i = \omega \quad \text{for } i = n-j+1,\ldots,n$$

where $(n-j)\varepsilon = 1-(2\sum_{i=1}^{(n-j)/2} x_{2i} + j\omega)$. Then $y \in K_{sub}$ and y dom x via $\{1,\ldots,n-1\}_y|\{2,\ldots,n\}_x$. <u>Case (ii)</u> j is odd: Define y by

$$y_i = y_{i+1} = x_{i+1} + \varepsilon \quad \text{for } i = 1,3,\ldots,n-j-2, \text{ and}$$

$$y_i = \omega \quad \text{for } i = n-j,\ldots,n$$

where $(n-j-1)\varepsilon = 1 - (2\sum_{i=1}^{(n-(j+1))/2} x_{2i} + (j+1)\omega)$. Then $y \in K_{sub}$ and y dom x via $\{1,\ldots,n-1\}_y|\{2,\ldots,n\}_x$.

<u>Necessity</u>: <u>Case (i)</u> j is even: Assume $\sum_{i=1}^{(n-j)/2} x_{2i} \geq (1-j\omega)/2$ and $x \in \mathrm{Dom}(K_{sub})$. Then there is some $y \in K_{sub}$ such that y dom x via $S_y|\{2,\ldots,n\}_x$. Since $\sum_{i=1}^{(n-j)/2} x_{2i} > (1-j\omega)/2$ and $y \in K_{sub} \subseteq A_0$, we obtain the contradiction $\sum_{i=1}^{n} y_i > 1$. <u>Case (ii)</u> j is odd: Assume $\sum_{i=1}^{(n-(j+1))/2} x_{2i} \geq (1-(j+1)\omega)/2$. Let $y \in K_{sub}$ dominate x. Then we get the contradiction $\sum_{i=1}^{n} y_i > 1$ since $\sum_{i=1}^{(n-(j+1))/2} x_{2i} \geq (1-(j+1)\omega)/2$ and $y \in A_0$.

When n is odd, the proof is similar to that above. □

From Claim 4, we obtain that for $j = 1,\ldots,n-2$, $\mathrm{Dom}_j(K_{sub}) = \mathrm{Dom}(K_{sub}) \cap A_j$

$$= \begin{cases} \left\langle \{x \in [A_j] \mid \sum_{i=1}^{(n-j)/2} x_{2i} < (1-j\omega)/2\} \right\rangle & \text{if } n \text{ and } j \text{ are even or } n \text{ and } j \text{ are odd} \\ \left\langle \{x \in [A_j] \mid \sum_{i=1}^{(n-(j+1))/2} x_{2i} < (1-(j+1)\omega)/2\} \right\rangle & \text{if } n \text{ is even, } j \text{ is odd or } n \text{ is odd and } j \text{ is even} \end{cases}$$

and $U_j(K_{sub}) = A_j - Dom_j(K_{sub})$.

Claim 5: If $x \in K_{sub}$, then there is no $y \in \bigcup_{j=1}^{n-1} A_j$ such that y dom x.

Proof of Claim 5: This is obvious since $x_n \geq \omega$ for any $x \in K_{sub}$ and $y_n < \omega$ for any $y \in \bigcup_{j=1}^{n-1} A_j$. □

Now we are ready to show that $K_{sub} = U^2(K_{sub})$. First, we note that $K_{sub} \subseteq U^2(K_{sub})$ follows from Claims 2 and 5. In order to show the converse, take any $x \in U(K_{sub}) - K_{sub}$. Then $x \in U_j(K_{sub})$ for some $j = 1,\ldots,n-1$ since $K_{sub} \subseteq A_0$ and $A_0 = K_{sub} \cup Dom(K_{sub})$. Assume that both n and j are even or odd. Then $\sum_{i=1}^{(n-j)/2} x_{2i} \geq (1-j\omega)/2$. If there is some $i(0) \in \{1,3,\ldots,n-j-1\}$ with $x_{i(0)} > x_{i(0)+1}$, then we can take some $y \in U_j(K_{sub})$ which dominates x. Thus we assume $x_i = x_{i+1}$ for all $i = 1,3,\ldots,n-j-1$. Then $\sum_{i=1}^{(n-j)/2} x_{2i} > (1-j\omega)/2$ since $x_i < \omega$ for all $i = n-j+1,\ldots,n$. Here the following two cases must be considered: (i) There is some $i \in \{2,4,\ldots,n-j-2\}$ with $x_i > x_{i+1}$; and (ii) $x_{n-j} > \omega$. In either case we can pick some $y \in U_j(K_{sub})$ which dominates x. When n is even and j is odd, or n is odd and j is even, we must have $\sum_{i=1}^{(n-(j+1))/2} x_{2i} \geq (1-(j+1)\omega)/2$. Thus in a manner somewhat similar to that above we can show that there is some $y \in U_j(K_{sub})$ which dominates x. Therefore we have shown $U^2(K_{sub}) \subseteq K_{sub}$. □

7.5 Cores and Supercores

Theorem 7.5: Consider any symmetric game (n,v) with $C \neq \emptyset$. Then C itself is a subsolution.

<u>Proof</u>: To simplify the notation, we will assume that the coordinates of any imputation x are arranged into nondecreasing order, i.e., $x_1 \leq x_2 \leq \dots \leq x_n$. Now let us begin the proof. Assume C is not a subsolution, i.e., $C \subsetneq U^2(C)$. Take any $x \in U^2(C) - C$. The following two claims are easily verified.

<u>Claim 1</u>: If y dom x, then $y \in \mathrm{Dom}(C)$.

<u>Claim 2</u>: There must exist some $i(0) \in \{1,\dots,n-1\}$ with $x_{i(0)} < x_{i(0)+1}$.

Let $t = \max\{i \in \{1,\dots,n-1\} \mid x_i < x_{i+1}\}$ and classify cases as follows.

<u>Case (i)</u> $\sum_{i=1}^{t+1} x_i < v(t+1)$: In this case, we must have $x_{t+1} = \dots = x_n > (1-v(t+1))/(n-(t+1))$ since $\sum_{i=t+2}^{n} x_i > 1-v(t+1)$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i \neq t+1 \\ x_{t+1} - (n-1)\varepsilon & \text{for } i = t+1 \end{cases}$$

where $0 < \varepsilon < \min((x_{t+1}-x_t)/n,\ (v(t+1) - \sum_{i=1}^{t+1} x_i)/(t+1),$ $(x_{t+1} - (1-v(t+1))/(n-(t+1)))/(n-1))$. Then $y \notin C$ and y dom x via $\{1,\dots, t, t+2\}_y \mid \{1,\dots,t,t+2\}_x$. $y \notin C$ is trivial since $\sum_{i=1}^{t+1} y_i = \sum_{i=1}^{t+1} x_i + t\varepsilon - (n-1)\varepsilon \leq \sum_{i=1}^{t+1} x_i < v(t+1)$. Now we will show that y dom x via $\{1,\dots,t,t+2\}_y \mid \{1,\dots,t,t+2\}_x$. From the definition of y, it is sufficient to show the effectiveness of $\{1,\dots,t,t+2\}_y$. Assume $\sum_{i=1}^{t} y_i + y_{t+2} > v(t+1)$. Then from the definition of ε, we obtain the contradiction

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{t} y_i + y_{t+2} + y_{t+1} + \sum_{i=t+3}^{n} y_i > v(t+1) + 1-v(t+1) = 1.$$

Hence there must exist some $z \in C$ such that z dom y via $\{1,\ldots,s\}_z|\{1,\ldots,s\}_y$. If $s \le t$, then z dom x via $\{1,\ldots,s\}_z|\{1,\ldots,s\}_x$ since $y_i > x_i$ for all $i = 1,\ldots,t$. This contradicts $x \in U^2(C)$. If $s \ge t+1$, then $z_{t+1} > y_{t+1} > x_{t+1} - (n-1)\varepsilon \ge (1-v(t+1))/(n-t-1)$ and thus we must have $\sum_{i=t+2}^{n} z_i > 1-v(t+1)$. On the other hand $\sum_{i=1}^{t+1} z_i \ge v(t+1)$ since $z \in C$. Therefore we obtain the contradiction $\sum_{i=1}^{n} z_i > 1$.

<u>Case (ii)</u> $\sum_{i=1}^{t+1} x_i \ge v(t+1)$: Since $x \notin C$, there must exist some $r \in \{1,\ldots,n-1\}$ such that $\sum_{i=1}^{r} x_i < v(r)$.

<u>(ii-I)</u> $\sum_{i=1}^{s} x_i \ge v(s)$ for all $s = t+2,\ldots,n$: In this case, $r \in \{1,\ldots,t\}$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i = 1,\ldots,t \\ x_{t+1} - t\varepsilon & \text{for } i = t+1 \\ x_i & \text{for } i = t+2,\ldots,n \end{cases}$$

where $0 < \varepsilon < \min((x_{t+1}-x_t)/n,\ (v(r) - \sum_{i=1}^{r} x_i)/r)$. Then $y \notin C$ and y dom x via $\{1,\ldots,r\}_y|\{1,\ldots,r\}_x$. Hence there is some $z \in C$ such that z dom y via $\{1,\ldots,u\}_z|\{1,\ldots,u\}_y$. Here we note that $u \le t$. In fact, for any $s > t$,

$$\sum_{i=1}^{s} y_i = \sum_{i=1}^{t} y_i + y_{t+1} + \sum_{i=t+2}^{s} y_i = \sum_{i=1}^{t} x_i + t\varepsilon + x_{t-1} - t\varepsilon + \sum_{i=t+2}^{s} x_i$$

$$= \sum_{i=1}^{t+1} x_i + \sum_{i=t+2}^{s} x_i = \sum_{i=1}^{s} x_i \ge v(s).$$

Therefore z dom x via $\{1,\ldots,u\}_z|\{1,\ldots,u\}_x$ which contradicts $x \in U^2(C)$.

<u>(ii-II)</u> $\sum_{i=1}^{s} x_i < v(s)$ for some $s = t+2,\ldots,n$: Take one of these s and let it be $s(0)$. Then $\sum_{i=s(0)+1}^{n} x_i > 1-v(s(0))$ and thus $x_{t+1} = \ldots = x_n > (1-v(s(0)))/(n-s(0))$ since $s(0) > t+1$. Define y by

$$y_i = \begin{cases} x_i + \varepsilon & \text{for } i \neq t+1 \\ x_{t+1} - (n-1)\varepsilon & \text{for } i = t+1 \end{cases}$$

where $0 < \varepsilon < \min((x_{t+1}-x_t)/n, (v(s(0)) - \sum_{i=1}^{s(0)} x_i)/s(0), (x_{t+1} - (1-v(s(0)))/(n-s(0)))/(n-1))$. Then $y \notin C$ and y dom x via $\{1,\ldots,t,t+2,\ldots,s(0)+1\}_y|\{1,\ldots,t,t+2,\ldots,s(0)+1\}_x$. It suffices to show the effectiveness of $\{1,\ldots,t,t+2,\ldots,s(0)+1\}_y$. Suppose $\sum_{i=1}^{t} y_i + \sum_{i=t+2}^{s(0)+1} y_i > v(s(0))$, then we obtain the contradiction

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{t} y_i + \sum_{i=t+2}^{s(0)+1} y_i + y_{t+1} + \sum_{i=s(0)+2}^{n} y_i$$

$$> v(s(0)) + 1-v(s(0)) = 1.$$

Hence there is some $z \in C$ such that z dom y via $\{1,\ldots,u\}_z|\{1,\ldots,u\}_y$ for some $u \in \{2,\ldots,n-1\}$. If $u \leq t$, then z dom x via $\{1,\ldots,u\}_z|\{1,\ldots,u\}_x$ which contradicts $x \in U^2(C)$. If $u \geq t+1$, then we must have $z_{t+1} > y_{t+1} > (1-v(s(0)))/(n-s(0))$ and thus $\sum_{i=s(0)+1}^{n} z_i > 1-v(s(0))$. On the other hand $\sum_{i=1}^{s(0)} z_i \geq v(s(0))$, since $z \in C$. Therefore we obtain the contradiction $\sum_{i=1}^{n} z_i > 1$.

Thus we have shown that $C = U^2(C)$ which implies that C is a subsolution. □

CHAPTER VIII

UNSOLVED PROBLEMS

Finally, we will list the following unsolved problems which merit further study and are closely related to this work.

1. (n;k) games.

   (i) Existence of (symmetric) stable sets.

   (ii) Determination of (symmetric) stable sets. Especially, it is of interest from the viewpoint of application, to determine symmetric stable sets for (n;k) games (n=2k-1) with one-point cores and for (n;k) games (n=2k) with one-point cores. The former games, suggested by S. Hart, reflect a kind of majority voting rule, and the author conjectures that the study of the latter games will lead us to the determination of symmetric stable sets for all Hart games given in Chapter VI.

2. General symmetric games.

   (i) Existence of (symmetric) stable sets when cores are nonempty.

   (ii) Existence of (symmetric) stable sets when cores are empty.

In order to solve these existence problems for general symmetric games, the determination of (symmetric) stable sets for (n;k) games might prove useful, as one approach. Another approach, which may be promising, is to use the set theoretical concepts which were developed in Roth [30] in proving the existence of subsolutions.

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
Technical Report No. 387

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
STABLE SETS FOR SYMMETRIC, n-Person, COOPERATIVE GAMES

5. TYPE OF REPORT & PERIOD COVERED
Technical Report

6. PERFORMING ORG. REPORT NUMBER
Technical Report No. 387

7. AUTHOR(s)
Shigeo Muto

8. CONTRACT OR GRANT NUMBER(s)
N00014-75-C-0678

9. PERFORMING ORGANIZATION NAME AND ADDRESS
School of Operations Research and Industrial Engineering, Cornell University, Upson Hall, Ithaca, New York 14853

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Operations Research Program
Office of Naval Research
Arlington, VA 22217

12. REPORT DATE
August 1978

13. NUMBER OF PAGES
136

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
n-person games, symmetric games, quota games, symmetric stable sets, semi-symmetric stable sets, systematic stable sets, subsolutions, cores, supercores.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Stable sets and subsolutions are studied mainly for symmetric, n-person, characteristic-function form games $(n;k)$ in which k-person coalitions are strongly vital, i.e., $v(s) \leq v(k)\cdot(s/k)$ for $k \leq s \leq n-1$ and $v(s) = 0$ for all $s < k$.

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In the first part, two types (i.e., systematic and semi-symmetric) of stable sets are defined and their existence is investigated. Furthermore, symmetric stable sets are determined for some classes of $(n;k)$ games.

In the latter half, the production game presented by S. Hart, which is a kind of $(n;k)$ game, is considered and his open questions are studied.

Finally, subsolutions defined by A. Roth are analyzed.

Our main results are summarized as follows.

1. Existence of systematic stable sets and determination of symmetric stable sets for $(n;k)$ games with $v(k) \leq 2/(n-k+2)$.
2. Existence of semi-symmetric stable sets for
   - (i) $(n;2)$ games
   - (ii) $(n;k)$ games ($n = qk + r$, $q \geq 2$ and $0 \leq r \leq k-1$) with one-point cores, and
   - (iii) $(n;k)$ games ($n = 2k-1$) with one-point cores.
3. (i) Determination of finite symmetric stable sets for $(n;k)$ games ($k \leq (n+1)/2$) with $v(k) \geq k/(n-k+1)$.
   - (ii) Uniqueness of such stable sets.
4. (i) Determination of symmetric stable sets for $(n;2)$, $(n;3)$ and $(n;4)$ games.
   - (ii) Their uniqueness for $(n;2)$ and $(n;3)$ games.
5. Uniqueness of Lucas' symmetric stable sets for $(n;n-1)$ games.
6. For Hart's production games $(n;k)_h$ ($n = qk + r$, $q \geq 2$ and $0 \leq r \leq k-1$), the following have been obtained.
   - (i) Determination of symmetric stable sets for
     - (a) $r = 0$ and $[[(k+1)/2]] \leq q \leq k-1$,
     - (b) $r \geq 1$ and $[[(k-r)/2]] \leq q \leq k-(r+2)$,
     - (c) $r = 0$, $k = 2\ell + 1$ ($\ell \geq 2$) and $q = \ell$, and
     - (d) $r = 0$, $k = 6$ and $q = 2$.
   - (ii) Conditions for the uniqueness of Hart's symmetric stable sets.
   - (iii) Existence of semi-symmetric stable sets.
7. For subsolutions, the following have been obtained.
   - (i) Determination of finite symmetric subsolutions for $(n;2)$ games.
   - (ii) Determination of symmetric subsolutions for $(n;n-1)$ games which are smaller than the symmetric stable sets presented by Lucas.
   - (iii) Coincidence of cores with supercores for general (not necessarily $(n;k)$) symmetric games.